Revision Notes

Class 12 Mathematics

Chapter 12 - Linear Programming

Linear Programming & Its Applications

- It is a common optimization (maximisation or minimization) approach used in business and everyday life to find the maximum or minimum values necessary of a linear expression in order to meet a set of supplied linear constraints.
- It may entail determining the most profit, the lowest cost, or the least amount of resources used, among other things.
- Industry, commerce, management science, and other fields use it.

Linear Programming Problem and its Mathematical Formulation

Optimal value:
Maximum or Minimum value of a linear function.

Objective Function:
- The function that needs to be improved (maximized/minimized)
- Linear function $Z = ax + by$, where a, b are constants, which has to be maximised or minimized is called a **linear objective function**.
- For example, $Z = 250x + 75y$ where variables x and y are called **decision variables.**

Linear Constraints:
- The objective function is to be optimised using a system of linear inequations/equations.
- In a linear programming issue, linear inequalities/equations or limitations on the variables are used.
- Also called **Overriding Conditions** or **Constraints.**
- The conditions $x \geq 0, y \geq 0$ are called **non-negative restrictions**.

Non-negative Restrictions:
All of the variables used to make decisions are assumed to have non-negative values.

Optimization problem:
- A problem that seeks to maximize or minimize a linear function (say of two variables x and y) subject to certain constraints as determined by a set of linear

inequalities.

- Linear programming problems (LPP) are a special type of optimization problem.

Note:

- The term "**linear**" denotes that all of the mathematical relationships in the problem are linear.
- The term "**programming**" refers to the process of deciding on a specific program or course of action.

Mathematical Formulation of the Problem

- A general LPP can be stated as

$(\text{Max} / \text{Min})\ Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

(Objective function) subject to constraints and non-negative restrictions.

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\le \ = \ \ge) b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\le \ = \ \ge) b_2 \\ . \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\le \ = \ \ge) b_m \end{array}\right\}$$

- Where
- $x_1, x_2, \ldots\ldots, x_n \ge 0$ where $a_{11}, a_{12}, \ldots., a_{mn}$;
- $b_1, b_2, \ldots., b_m$ and $c_1, c_2, \ldots., c_n$ are **constants** and
- $x_1, x_2, \ldots\ldots, x_n$ are **variables.**

Graphical method of solving linear programming problems
Terminologies

Solution of an LPP:
A set of values of the variables $x_1, x_2, \ldots\ldots, x_n$ that satisfy the restrictions of an LPP.

Feasible Solution of an LPP:

- A set of values of the variables $x_1, x_2, \ldots\ldots, x_n$ hat satisfy the restrictions and **non-negative restrictions of an LPP**.

- **Possible solutions to the restrictions** are represented as points within and on the boundary of the feasible zone.

Feasible Region:
The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem, is called **the feasible region** (or solution region) for the problem.

Feasible Choice:
Each point is in the feasible region.

Infeasible Region:
The region outside the feasible region.

Infeasible Solution:
Any point outside the feasible region.

Optimal Solution of an LPP:
A feasible solution of an LPP is said to be optimal (or optimum) if it also **optimizes the objective function** of the problem.

Graphical Solution of an LPP:
The solution of an LPP is obtained by the graphical method, that is by drawing the **graphs** corresponding to the **constraints and the non-negative restrictions.**

Unbounded Solution:
Such solutions exist if the value of the objective function can be **increased or decreased** forever.

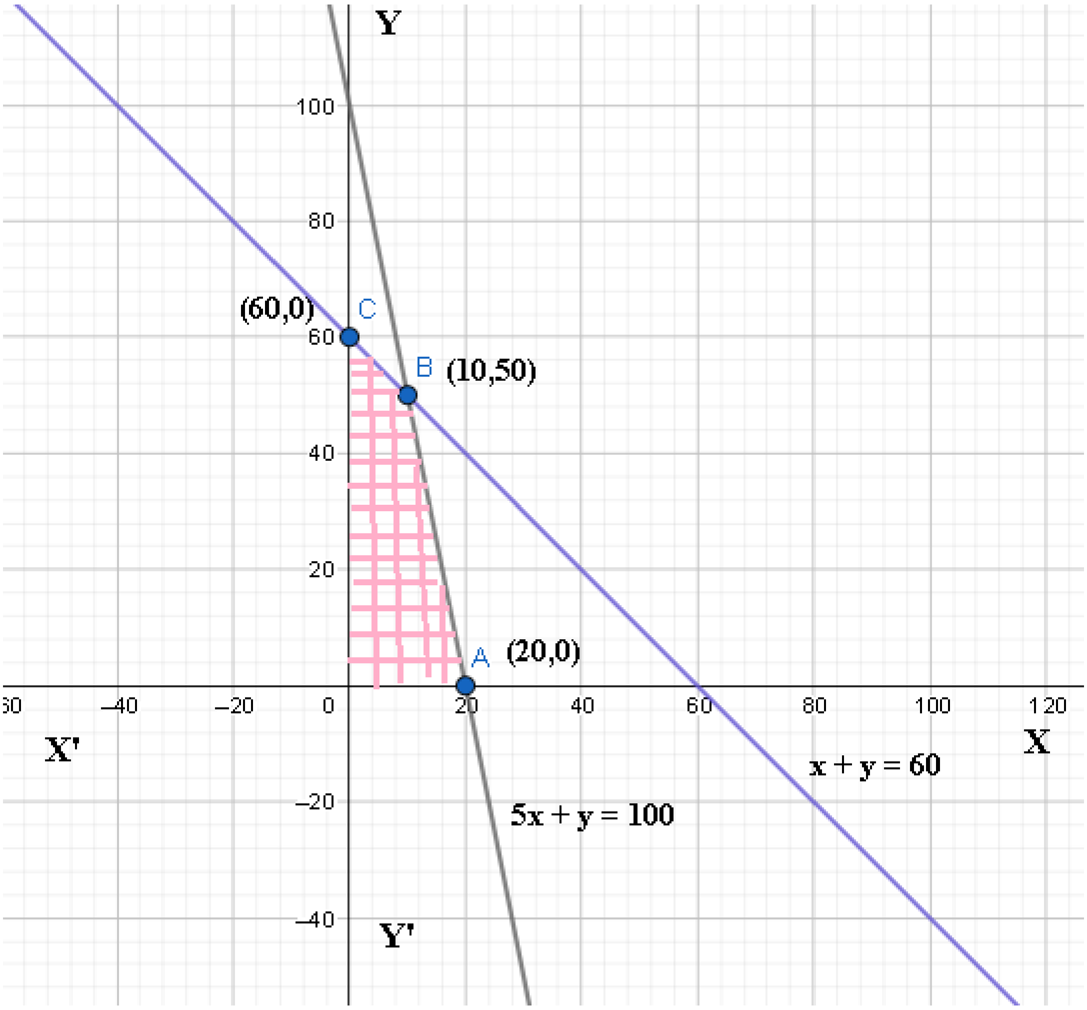
Example:
Graph the constraints stated as linear inequalities:
- $5x + y \leq 100$ ... (1)
- $x + y \leq 60$ ...... (2)
- $x \geq 0$ .... (3)
- $y \geq 0$ .... (4)

Solution:
For Plotting the Equation (1),
- Let $x = 0$. Hence we get the point $y = 100$

- Let $y = 0$. Hence we get the point $x = \frac{100}{5} = 20$
- The equation (1) is obtained by joining the points $(20,100)$

For Plotting the Equation (2),

- Let $x = 0$. Hence we get the point $y = 60$
- Let $y = 0$. Hence we get the point $x = 60$
- The equation (2) is obtained by joining the points $(60,60)$
- From Equation (3) and Equation (4), we know both x and y are more significant than 0.



From the graph above,

- Possible solutions to the constraints are represented by points within and on the edge of the feasible zone.
- Here, every point within and on the boundary of the feasible region OABC represents a possible solution to the problem.

- For example, point $(10,50)$ is a feasible solution to the problem, and so are the points $(0,60),(20,0)$, etc.
- Any point outside the possible region is called an infeasible solution. For example, point $(25,40)$ is an infeasible solution to the problem.
- Now, we can see that every point in the feasible region $OABC$ satisfies all the constraints given in (1) to (4). Because there are endless points, it is unclear how to discover a position that yields the objective function's most significant value $z = 250x + 75y$.

Vertex of the Feasible Region	The corresponding value of Z (in Rs)
O $(0,0)$	0
A $(0,60)$	4500
B $(10,50)$	6250 (Maximum)
C $(20,0)$	5000

Theorem 1

- Let R be the feasible region (convex polygon) for a linear programming problem.
- Let $Z = ax + by$ be the objective function.
- When the variables x and y are subject to constraints specified by linear inequalities, the optimal value Z (maximum or minimum) must occur at a **corner point* (vertex) of the feasible region.**

Theorem 2

- Let R be the feasible region for a linear programming problem.
- Let $Z = ax + by$ be the objective function.
- If an objective function R is bounded**, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.
- **Remark:**

The objective function may not have a **maximum or minimum** value if R is **unbounded**. It must, however, occur at a corner point of R if it exists.

A Graphical Approach to Solving a Linear Programming Issue

It can be solved by using below methods, they are as follows;

1. Corner point method

2. Iso-profit or Iso-cost method

Corner Point Method:

- Based on the principle of the extreme point theorem.
- Procedure to Solve an LPP Graphically by Corner Point Method
- Consider each constraint as an equation.
- Plot each equation on a graph, as each one will geometrically represent a straight line.
- The common region thus obtained satisfying all the constraints, and the **non-negative restrictions** are called the **feasible region**. It is a **convex polygon.**
- Determine the **vertices** (corner points) **of the convex polygon**. These vertices are known as the feasible region's **extreme points of corners**.
- Find the objective function's values at each of the extreme points.
- The optimal solution of the given LPP is the point where the value of the objective function is optimum (maximum or minimum).

Isom-profit or Iso-cost Method:

Iso-profit or Iso-cost Method for Graphically Solving an LPP:

- Consider each constraint to be a mathematical equation.
- Draw each equation on a graph, as each one will represent a straight line geometrically.
- The polygonal region reached by meeting all constraints and non-negative limits is the feasible region, a convex set of all viable solutions of a given LPP.
- Determine the viable region's extreme points.
- Give the objective function Z a handy value k and draw the matching straight line in the xy-plane.
- If the problem is maximization, draw parallel lines to $Z = k$ and find the line farthest from the origin and has at least one point in common with the viable zone.
- If the problem is of minimisation, then draw lines parallel to the line $Z = k$ that is closest to the origin and has minimum one point in common with the feasible zone.
- The optimal solution of the given LPP is the common point so achieved.

Working Rule for Marking Feasible Region

Consider the constraint $ax + by \leq c$, where $c > 0$.

- To begin, make a straight line $ax + by = c$ y connecting any two points on it.
- Find two points that satisfy this equation as a starting point.
- This straight-line divides the xy -plane into two parts.
- The inequation $ax + by\ c$ will represent that part of the xy -plane which lies to that side of the line $ax + by = c$ in which the origin lies.

Again, consider the constraint $ax + by \geq c$, where $c > 0$.

- Draw the straight line $ax + by = c$ by joining any two points on it.

- This straight-line divides the xy-plane into two parts.
- The inequation $ax + by \geq c$ will represent that part of the xy-plane, which lies to that side of the line $ax + by = c$ in which the origin does not lie.

Important Points to be remembered

1. Basic Feasible Solution:

A fundamental solution that also meets the **non-negativity** constraints is known as a **BFS**.

2. Optimum Basic Feasible Solution:

A BFS is said to be optimum if it also o**ptimizes** (Max or min) the objective function.

Example:

Solve the following linear programming problem graphically:

Maximize $Z = 4x + y \ldots (1)$

Subject to the constraints:

- $x + y \leq 50 \ \ldots (2)$
- $3x + y \leq 90 \ \ldots (3)$
- $x \geq 0,\ y \geq 0 \ldots (4)$

Solution:

For Plotting the Equation (2),

- Let $x = 0$, Hence we get the point $y = 50$
- Let $y = 0$. Hence we get the point $x = 50$
- Equation (2) is obtained by joining the points $(50,50)$

For Plotting the Equation (3),

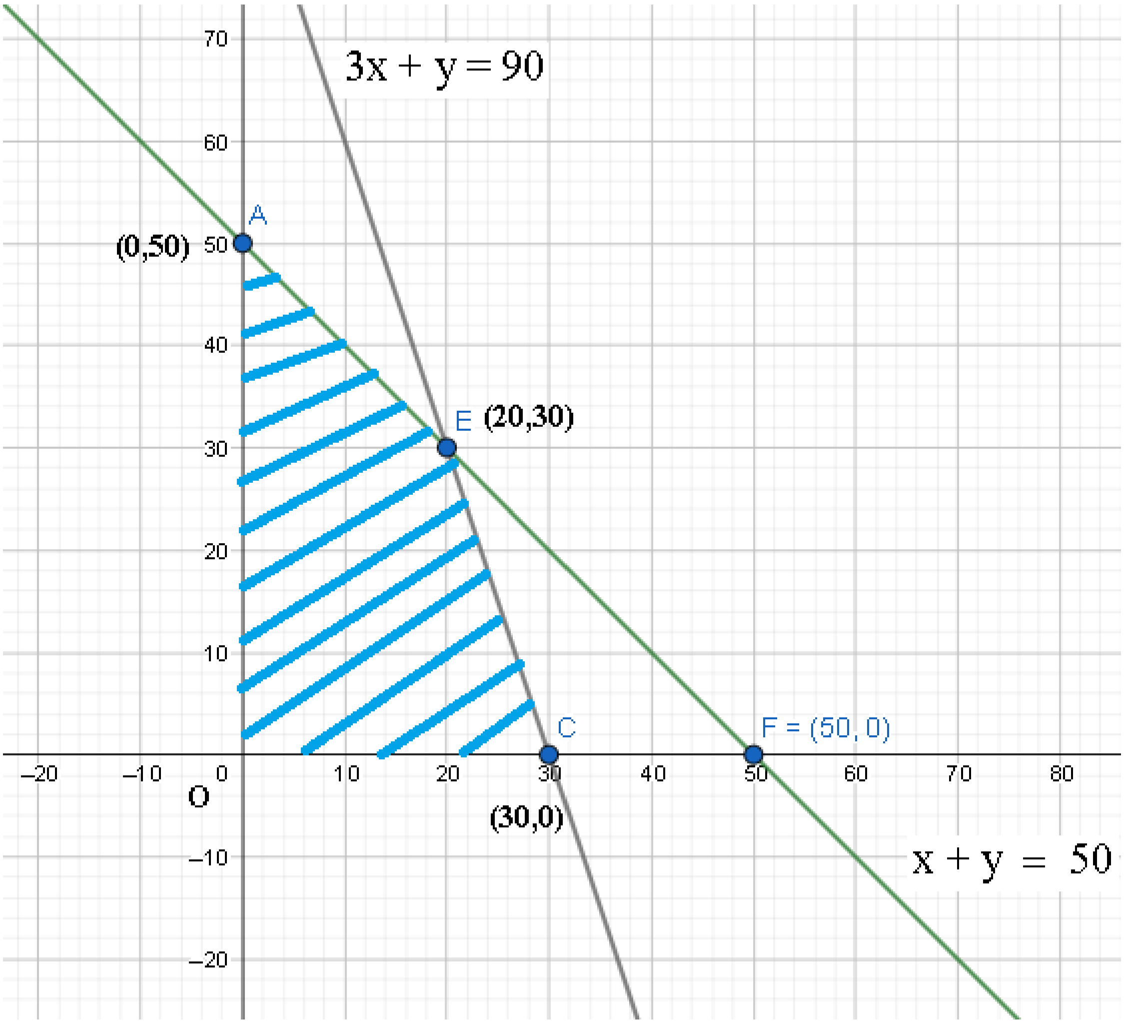
- Let $x = 0$. Hence we get the point $y = 90$
- Let $y = 0$. Hence we get the point $x = 30$
- The equation (3) is obtained by joining the points $(30,90)$

From Equation (4), we know both x and y are greater than 0.

As a result, the points are $(0,0)$, $(50,50)$, $(0,50)$, $(30,0)$, $(30,90)$.

The viable region in the graph is colored, as determined by the system of constraints (2) to (4).

The viable region $OAEC$ is bounded, as shown below;



By replacing the vertices of the bounded region for the vertices of the bounded region, the maximum value of Z may be determined using the **Corner Point Method**.

As a result, the highest value of Z at the position is 120 at the point $(30,0)$.

Corner Point	The corresponding value of Z
$(0,0)$	0
$(30,0)$	120 (Maximum)
$(20,30)$	110
$(0,50)$	50

Example:

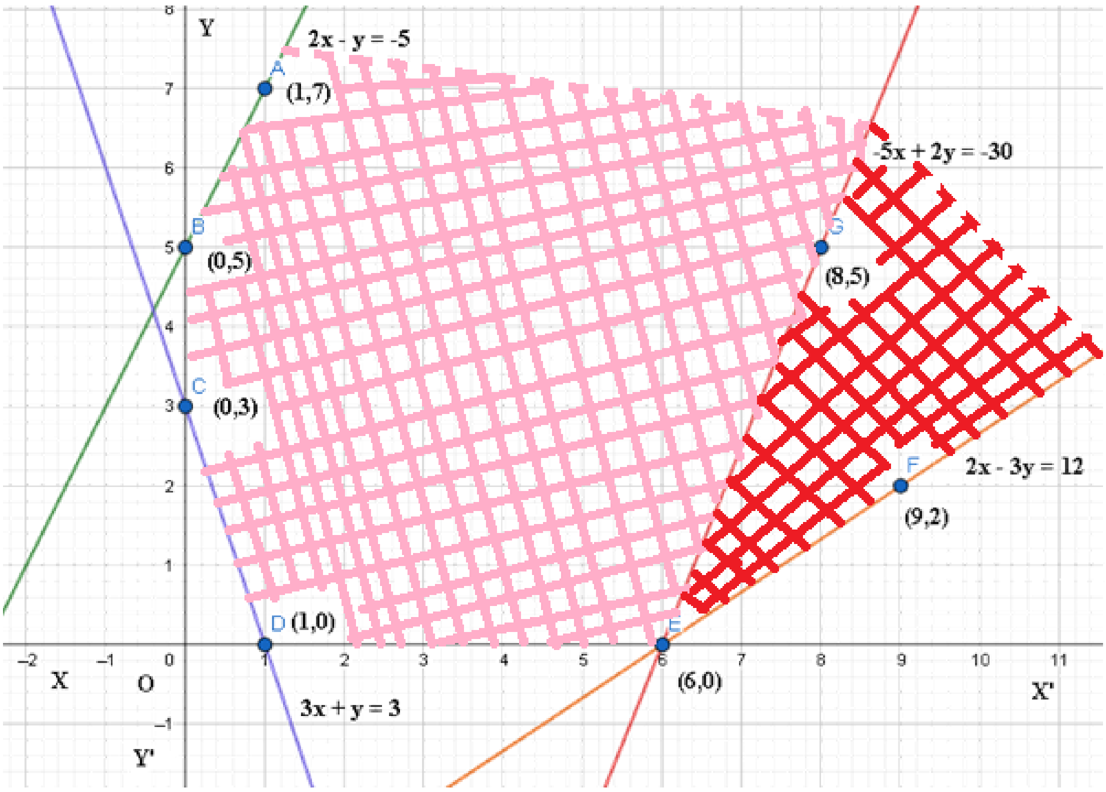
Determine the minimum value of the objective function graphically.,

$Z = -50x + 20y$ ... (1)

Subject to the constraints:

- $2x - y \geq -5 \ldots (2)$
- $3x + y \geq 3 \ldots (3)$
- $2x - 3y \leq 12 \ldots (4)$
- $x \geq 0, y \geq 0 \ldots (5)$

Solution:
We need to graph the feasible region of the system of inequalities (2) to (5). The viable shaded area is shown in the graph.



Corner Point	$Z = -50x + 20y$
$(0,5)$	100
$(0,3)$	60
$(1,0)$	-50
$(6,0)$	-300 (Smallest)

By Observation that the feasible region is **unbounded**.
At the corner points, we now examine Z.
From this table, we found that -300 is the smallest value of Z at the corner point

$(6,0)$.

Since the region would have been bounded, this smallest value of Z is the minimum value of Z (Theorem (2)).

But here, we have seen that the feasible part is unbounded.
Therefore, -300 may or may not be the minimum value of Z.
We use a graph to decide on this topic.
$-50x + 20y < -300$ that is,
$-5x + 2y < -30$ (By dividing the above Equation by 10)

And need to confirm whether the resulting open half-plane has points in common with the feasible region or not.
If it has common attributes, then -300 will not be the minimum value of Z. Otherwise, -300 will be the minimum value of Z.
As shown in the above graph, it has common points, and hence,
$Z = -50x + 20y$ has **no minimum value** subject to the given constraints.

General features of linear programming problems

1. A **convex** region is always the **viable zone**.
2. The **vertex (corner)** of the feasible region is where the objective function's **maximum** (or most minor) solution occurs.
3. If two corner points have the same **maximum** (or lowest) objective function value, then every point on the line segment connecting them has the same **top** (or minimum) value.

Different Types of Linear Programming Problems:

The following are some of the most crucial linear programming problems:

1. Manufacturing problems:

When each product requires a fixed quantity of workforce, machine hours, labour hour per unit of product, warehouse space per unit of output, and so on, determine the number of units of various things that a company should manufacture and sell to optimise profit.

2. Diet problems:

Determine the number of constituents/nutrients that should be included in a diet to keep the expense of the intended diet as low as possible while ensuring that each constituent/nutrient is present in a minimum amount.

3. Transportation problems:

Determine a transportation timetable to determine the most cost-effective method of moving a product from various plants/factories to multiple to keep the intended diet’s expense markets.