Revision Notes

Class 12 Maths

Chapter 10 – Vector Algebra

Vector

Vector quantities are those quantities that have **magnitude and direction**. It is generally represented by a directed line segment. We represent a vector as $\overrightarrow{AB}$, where **initial point** of vector is denoted by A and the **terminal point** by B. The magnitude of vector is expressed as $\left|\overrightarrow{AB}\right|$.

Position Vector

Let us denote the origin as O such that this is a fixed point. There is a point, say P at a distance from O. Now, the position vector of a point P is given by the vector $\overrightarrow{OP}$.

The next case is when there are two vectors, $\vec{a}$ and $\vec{b}$ which represent the position vectors of two points A and B. Then we can write the vector $\overrightarrow{AB}=\vec{b}-\vec{a}$ or the position vector of $B-$ the position vector of A.

Types of vectors

1. **Zero Vector** – It has **zero magnitude**. This means that vector has the same initial and terminal point. It is denoted by $\vec{O}$. The direction of zero vector is indeterminate.

2. **Unit Vector** – It has **unit magnitude**. Unit vector in direction of a vector $\vec{a}$ is denoted by $\hat{a}$ and symbolically as $\hat{a}=\frac{\vec{a}}{\left|\vec{a}\right|}$.

3. **Co-initial Vectors** - Two or more vectors are said to be co-initial if they have the **same initial point**.

4. **Equal Vectors** - Two vectors are said to be equal if they have the **same magnitude and direction**. They represent the same physical quantity.

5. **Collinear Vectors** – Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their direction. For this reason, they are also called **parallel vectors**. We have two sub-categories – like

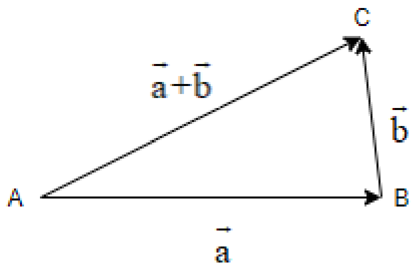
vectors (same direction) and unlike vectors (different directions). We can represent it mathematically by taking two non-zero vectors $\vec{a}$ and $\vec{b}$. They are collinear if and only, if $\vec{a}=K\vec{b}$, where $K \in R-\{0\}$.

6. **Coplanar Vectors** – Those vectors which lie on the same plane and they are all parallel to the same plane. We must remember that **two vectors are always coplanar**.

7. **Negative Vector** – A vector which has same magnitude but opposite direction to another vector is called negative of that vector.

Addition of vectors

1. **Triangle Law** – Consider a triangle ABC. Let the sum of two vectors $\vec{a}$ and $\vec{b}$ be represented by $\vec{c}$. The position vectors are represented by $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.



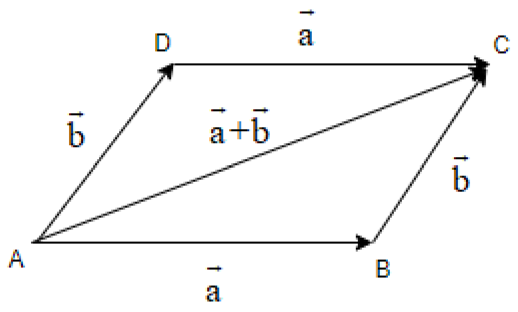
Triangle law of vector addition states that when two vectors are represented as two sides of the triangle with the order of magnitude and direction, then the third side of the triangle represents the magnitude and direction of the resultant vector. So, we can write that $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$.

2. **Parallelogram Law** – Consider a parallelogram $ABCD$. Let the sum of two vectors $\vec{a}$ and $\vec{b}$ be represented by $\vec{c}$. The position vectors are represented as

$\vec{a}=\overrightarrow{AB}=\overrightarrow{DC}$

$\vec{b}=\overrightarrow{AD}=\overrightarrow{BC}$

$\vec{a}+\vec{b}=\overrightarrow{AC}$



According to the parallelogram law of vector addition if two vectors act along two adjacent sides of a parallelogram (having magnitude equal to the length of the sides) both pointing away from the common vertex, then the resultant is represented by the diagonal of the parallelogram passing through the same common vertex and in the same sense as the two vectors.

The sum is

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$$

3. Properties of vector addition

a) Commutative property - $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

b) Associative property - $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

c) Zero is the additive identity - $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$

d) $\vec{a}+\left(-\vec{a}\right)=\vec{0}=\left(-\vec{a}\right)+\vec{a}$

Multiplication of a vector by a scalar

If $\vec{a}$ is a vector and m is a scalar, then their product is $m\vec{a}$. The magnitude would be $|m|$ times the magnitude of $\vec{a}$. This is called **scalar multiplication**. If $\vec{a}$ and $\vec{b}$ are vectors and m and n are scalars, then

a) $m\left(\vec{a}\right)=\left(\vec{a}\right)m=m\vec{a}$

b) $m\left(n\vec{a}\right)=n\left(m\vec{a}\right)=(mn)\vec{a}$

c) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

d) $m\left(\vec{a}+\vec{b}\right)=m\vec{a}+m\vec{b}$

Component form of vectors

- We have to consider three axis - x, y, z and a point in the coordinate axis. So, the position vector for such a point would be written as $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$. This is the component form of vector.
- The **scalar components are** x, y, z **and the vector components are** $x\hat{i}, y\hat{j}, z\hat{k}$**.**
- Consider two vectors as $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{B} = p\hat{i} + q\hat{j} + r\hat{k}$, then
  a) Sum is given by $\vec{A} + \vec{B} = (a+p)\hat{i} + (b+q)\hat{j} + (c+r)\hat{k}$.
  b) Difference is given by $\vec{A} - \vec{B} = (a-p)\hat{i} + (b-q)\hat{j} + (c-r)\hat{k}$.
  c) Multiplication by a scalar m is given by $m\vec{A} = ma\hat{i} + mb\hat{j} + mc\hat{k}$.
  d) The vectors are equal if $a = p, b = q, c = r$.

Test for collinearity

Three points A,B,C with position vectors $\vec{a},\vec{b},\vec{c}$ respectively are collinear, if and only if there exist scalar x,y,z not all zero simultaneously such that; $x\vec{a}+y\vec{b}+z\vec{c}=0$, where $x+y+z=0$.

Test for coplanar points

Four points A,B,C,D with position vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ respectively are coplanar if and only if there exist scalars x,y,z,w not all zero simultaneously such that; $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$, where $x+y+z+w=0$.

Section formula

a) Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A and B. A point R with position vector as $\vec{r}$ divides $\overrightarrow{AB}$ such that $m\overrightarrow{RB} = n\overrightarrow{AR}$ and this denotes that $\overrightarrow{AB}$ is divided **internally** in the ratio $m:n$ is given by $\vec{r}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$.

b) Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A and B. A point R with position vector as $\vec{r}$ divides $\overrightarrow{AB}$ such that $m\overrightarrow{RB} = n\overrightarrow{AR}$ and this denotes that $\overrightarrow{AB}$ is divided **externally** in the ratio $m:n$ is given by

$\vec{r}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$.

c) Now if the ratio is 1:1, then we can obtain the position vector of the **midpoint** as $\dfrac{\vec{a}+\vec{b}}{2}$.

Magnitude of vector

a) For a vector $\vec{A}=a\hat{i}+b\hat{j}+c\hat{k}$, magnitude is $|A|=\sqrt{a^2+b^2+c^2}$.

b) For vector $\overrightarrow{AB}$ with $\vec{A}=a\hat{i}+b\hat{j}+c\hat{k}$ and $\vec{B}=p\hat{i}+q\hat{j}+r\hat{k}$, the magnitude is $\left|\overrightarrow{AB}\right|=\sqrt{(p-a)^2+(q-b)^2+(r-c)^2}$.

Product of vectors

1. Scalar Product

- It is also called **dot product**. For two vectors $\vec{a}$ and $\vec{b}$, the dot product can be represented as $\vec{a}.\vec{b}$ and it is defined as $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta;(0\le\theta\le\pi)$.
- From this, we can find the angle between vectors as $\cos\theta=\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$.
- We have the below possibilities:

a) If θ is acute, then $\vec{a}\cdot\vec{b}>0$.

b) If θ is obtuse, then $\vec{a}\cdot\vec{b}<0$.

c) If θ is zero, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$.

d) If θ is π, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$.

- If vectors $\vec{a}$ and $\vec{b}$ are non-zero and $\vec{a}.\vec{b}=0$, then it is the condition for them to be **perpendicular vectors**.
- Considering component form and above point, we get results as

  a) $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=k\cdot k=1$

  b) $\hat{i}\cdot\hat{j}=\hat{j}\cdot k=k\cdot\hat{i}=0$

- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3k$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3k$ then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$.
- **Properties of scalar product**

a) $\vec{a}\cdot\vec{a}=|\vec{a}|^2=\vec{a}^2, \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Commutative)

b) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive)

c) $(m\vec{a})\cdot\vec{b}=\vec{a}\cdot(m\vec{b})=m(\vec{a}\cdot\vec{b})$ (Associative), where m is scalar.

- **Projection of vector** $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.
- Maximum value of $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$
- Minimum value of $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$
- A vector in the direction of the bisector of the angle between two vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$.
- Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda\in R^+$.
- Bisector of the exterior angle between $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a}-\hat{b})\lambda\in R-\{0\}$.

2. Vector Product

- It is also called **cross product**. For two vectors $\vec{a}$ and $\vec{b}$, the vector product is represented as $\vec{a}\times\vec{b}$ and is defined by $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$, where θ is the angle between them and $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed screw system.
- From this, we can write the angle between vectors as $\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.
- If vectors $\vec{a}$ and $\vec{b}$ are non-zero and $\vec{a}\times\vec{b}=0$, then it is the condition for them to be **parallel vectors**.
- Considering component form and above point, we get results as

  a) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$

b) $\hat{i}\times\hat{j}=k, \hat{j}\times k=\hat{i}, k\times\hat{i}=\hat{j}$

- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3k$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3k$ then $\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & k \\ a_1 & a_2 & a \\ b_1 & b_2 & b \end{vmatrix}$.
- Geometrically, we can define $\left|\vec{a}\times\vec{b}\right|=$ area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.
- **Properties of vector product**

  a) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (Not Commutative)

  b) $\left(m\vec{a}\right)\times\vec{b}=\vec{a}\times\left(m\vec{b}\right)=m\left(\vec{a}\times\vec{b}\right)$ (Associative) where m is scalar.

  c) $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)$ (Distributive)

- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is

  $$n=\pm\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$$

- A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is

  $$\pm\frac{r\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}$$

- If $\vec{a},\vec{b}$ and $\vec{c}$ are the position vectors of vertices A, B and C of a triangle, then the vector **area of triangle** is given by

  $$ABC=\frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$$

  The points A, B and C are **collinear** if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$.

- **Area of quadrilateral** whose diagonal vectors are $\vec{d_1}$ and $\vec{d_2}$ is given by $\frac{1}{2}\left|\vec{d_1}\times\vec{d_2}\right|$.

Scalar Triple Product

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\vec{a}.\left(\vec{b}\times\vec{c}\right)$ and can be represented as $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$. It is also referred to as **box product**.
- Geometrically, it represents the **volume of the parallelepiped** whose three coterminous edges are represented by $\vec{a}, \vec{b}$ and $\vec{c}$. So $V = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$.
- Scalar triple product is **cyclic**, i.e. the order of vectors can be interchanged in a cyclic manner as shown below,

  $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ or $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{c}\ \vec{a}\ \vec{b}\right]$

  $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = -\vec{a}\cdot\left(\vec{c}\times\vec{b}\right)$ or $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = -\left[\vec{a}\ \vec{c}\ \vec{b}\right]$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3 k$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3 k$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3 k$ then

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a \\ b_1 & b_2 & b \\ c_1 & c_2 & c \end{vmatrix}$$

- Scalar product of three vectors, two of which are equal or parallel is 0.
- Vectors $\vec{a}, \vec{b}, \vec{c}$ are **coplanar** if $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.