# Revision Notes

# Class 12 Mathematics

# Chapter 9 – Differential Equations

- **Definition:**

An equation involving the dependent variable and independent variable and also the derivatives of the dependable variable is known as differential equation. This can be mathematically written as $x\frac{dy}{dx}+y=0$.

The derivative $\frac{dy}{dx}$ can also be written as $f'(x)$ or $y'(x)$. Similarly,

$\frac{d^2y}{dx^2} \Rightarrow f''(x)$ or $y''(x)$

$\frac{d^3y}{dx^3} \Rightarrow f'''(x)$ or $y'''(x)$

Some examples can be $\frac{dy}{dx}=\frac{x}{y^{\frac{1}{3}}\left(1+x^{\frac{1}{3}}\right)}$, $\frac{d^2y}{dx^2}=-p^2y$ or $x^2\left(\frac{dy}{dx}\right)^2=y^2+1$.

Differential equations which involve only one independent variable are called **ordinary differential equations**.

- **Order of Differential Equations:**

The order of a differential equation is the order of the highest derivative involved in the differential equation. This can be understood clearly by looking at few examples.

i. First order differential equation - $\left(\frac{dy}{dx}\right)^4+\left(\frac{dy}{dx}\right)^2-5x=0$. The maximum derivative of $y$ with respect to $x$ is $\frac{dy}{dx}$.

ii. Second order differential equation - $\frac{d^2y}{dx^2}+7y=0$. The maximum derivative of $y$ with respect to $x$ is $\frac{d^2y}{dx^2}$.

iii.Third order differential equation - $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right) + 2 = 0$ . The maximum derivative of $y$ with respect to $x$ is $\frac{d^3y}{dx^3}$.

- **Degree of Differential Equations:**

The degree of a differential equation is the degree of the highest differential coefficient when the equation has been made rational and integral as far as the differential coefficients are concerned. This can be understood clearly by looking at few examples.

i. First degree differential equation - $\frac{dy}{dx} = \frac{5x}{y^{\frac{1}{3}}\left(1 - x^{\frac{1}{3}}\right)}$. The power of the highest order derivative $\frac{dy}{dx}$ is $1$.

ii.Second degree differential equation - $\left(\frac{d^3y}{dx^3}\right)^2 + 6\left(\frac{dy}{dx}\right) = -2$ . The power of highest order derivative $\frac{d^3y}{dx^3}$ is $2$.

iii. Third degree differential equation - $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/3} = 3\frac{d^2y}{dx^2}$ . First, making it rational, $\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 27\left(\frac{d^2y}{dx^2}\right)^3$. The power of highest order derivative $\frac{d^2y}{dx^2}$ is $3$.

**Illustration 1: Find the order and degree of the following differential equations.**

**i.** $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$

**Ans:** Rewriting it as $\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$

So, the order $= 2$ and the degree $= 3$.

**ii.** $\mathbf{\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{5/3}}$

**Ans:** Rewriting it as $\left(\frac{d^2y}{dx^2}\right)^3 = \left[1 + \left(\frac{dy}{dx}\right)^4\right]^5$.

So, the order $= 2$ and the degree $= 3$.

**iii.** $\mathbf{y = px + \sqrt{a^2p^2 + b^2}}$ **where** $\mathbf{p = \frac{dy}{dx}}$

**Ans:** Substituting p and then rewriting it as $\left(y - x\frac{dy}{dx}\right)^2 = a^2\left(\frac{dy}{dx}\right)^2 + b^2$.

So, the order $= 1$ and the degree $= 2$.

- **Formation of Ordinary Differential Equation:**

There may be some arbitrary constants in an equation containing variables and constants. An ordinary differential equation is formed as a result of elimination of these arbitrary constants.

Consider an equation containing $n$ arbitrary constants. Differentiating this equation n times we get $n$ additional equations containing n arbitrary constants and derivatives. Eliminating $n$ arbitrary constants from the above $(n+1)$ equations, differential equation involving nth derivative is obtained. After this is complete, the resulting equation will be of the form $\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$

**Illustration 2: Find the differential equation of the family of all circles which pass through the origin and whose centre lie on $y-$axis.**

**Ans:** Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

If it passes through $(0,0)$, then $c = 0$

$\therefore$ The equation of circle is $x^2 + y^2 + 2gx + 2fy = 0$

Since the centre of the circle lies on $y-$axis then $g = 0$.

$\therefore$ The equation of the circle is

$x^2 + y^2 + 2fy = 0$......(i)

This represents family of circles.

Differentiating gives,

$$2x+2y\frac{dy}{dx}+2f\frac{dy}{dx}=0..........(ii)$$

From (i) and (ii),

$$\left(x^2+y^2\right)\frac{dy}{dx}-2xy=0$$

Hence, this is the required differential equation.

- **Solution of a Differential Equation:**

The solution of the differential equation is a relation is a relation between the independent and dependent variable free from derivatives satisfying the given differential equation.

So, the solution of an equation given by $\frac{dy}{dx}=m$ can be obtained by integrating both the sides to remove the derivatives and obtain $y=mx+c$, where $c$ is arbitrary constant.

**a) General Solution or Primitive**

The general solution of a differential equation is the relation between the variables (not involving the derivatives) which contain the same number of the arbitrary constants as the order of the differential equation.

Thus the general solution of the differential equation $\frac{d^2y}{dx^2}=4y$ is $y=A\sin 2x+B\cos 2x$, where $A$ and $B$ are the constants.

**b) Particular Solution or Integral**

A solution which is obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution.

**Illustration 3: Show that $v=\frac{A}{r}+B$ is the general solution of the second order differential equation $\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0$, where $A$ and $B$ are arbitrary constant.**

**Ans:** Given $v=\frac{A}{r}+B$

Differentiating, $\frac{dv}{dr}=-\frac{A}{r^2}$.

Differentiating again, $\frac{d^2v}{dr^2}=\frac{2A}{r^3}$.......(i)

Rearranging the second term and substituting first derivative,

$$\frac{d^2v}{dr^2} = -\frac{2}{r}\frac{dv}{dr}$$

$$\frac{2A}{r^3} = -\frac{2}{r}\left(-\frac{A}{r^2}\right)$$

$$\frac{2A}{r^3} - \frac{2A}{r^3} = 0$$

Putting $A = 4, B = 5$ in $v = \frac{A}{r} + B$ we get a particular solution of the differential equation

$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$ is $v = \frac{4}{r} + 5$.

**Illustration 4: Show that $y = ae^x + be^{2x} + ce^{-3x}$ is a solution of the equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$.**

**Ans:** Given that

$y = ae^x + be^{2x} + ce^{-3x}$ ...(i)

Differentiating,

$y' = ae^x + 2be^{2x} - 3ce^{-3x}$ ...(ii)

Differentiating (ii),

$y'' = ae^x + 4be^{2x} + 9ce^{-3x}$

Differentiating again,

$y''' = ae^x + 8be^{2x} - 27ce^{-3x}$

The given differential equation is $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$.

Considering the LHS and substituting the terms,

$\left\lfloor ae^x + 8be^{2x} - 27ce^{-3x} \right\rfloor - 7\left\lfloor ae^x + 2be^{2x} - 3ce^{-3x} \right\rfloor + 6\left\lfloor ae^x + be^{2x} + ce^{-3x} \right\rfloor$

$\Rightarrow ae^x + 8be^{2x} - 27ce^{-3x} - 7ae^x - 14be^{2x} + 21ce^{-3x} + 6ae^x + 6be^{2x} + 6ce^{-3x}$

$\Rightarrow 0$

This is equal to RHS.

Since it satisfies the equation, $y = ae^x + be^{2x} + ce^{-3x}$ is the solution for $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$.

- **Method of solving an equation of the first order and first degree:**

A differential equation of the first order and first degree can be written in the form $\frac{dy}{dx}=f\left(x,y\right)$ or, $Mdx+Ndy=0$, where $M$ and $N$ are functions of $x$ and $y$.

**1. Method – 1**

**i. Variable Separation:**

The general form of such an equation is

$f(x)dx+f(y)dy=0$ ...(i)

Integrating it gives the solution as

$\int f(x)dx+\int f(y)dy=c$

**ii. Solution of differential equation of the type $\frac{dy}{dx}=f\left(ax+by+c\right)$:**

Consider the differential equation $\frac{dy}{dx}=f\left(ax+by+c\right)$ ...(i) where $f\left(ax+by+c\right)$ is some function of $ax+by+c$.

Let $z=ax+by+c$

$\therefore \frac{dz}{dx}=a+b\frac{dy}{dx}$

or, $\frac{dy}{dx}=\frac{\frac{dz}{dx}-a}{b}$

From (i), $\frac{\frac{dz}{dx}-a}{b}=f(z)$

or, $\frac{dz}{dx}=bf(z)+a$

or, $\frac{dz}{bf(z)+a}=dx$ ...(ii)

In the differential equation (ii), the variables $x$ and $z$ are separated.

Integrating, we get

$\int\frac{dx}{bf(z)+a}=\int dx+c$

or, $\int\frac{dx}{bf(z)+a}=x+c$ , where $z=ax+by+c$

This represents the general solution of the differential equation (i)

**Illustration 5: Solve $(x-y)^2\frac{dy}{dx}=a^2$.**

**Ans:** Let $x-y=v$ and differentiate it to get

$$\Rightarrow \frac{dy}{dx}=1-\frac{dv}{dx}$$

Substituting these in $(x-y)^2\frac{dy}{dx}=a^2$ and rearranging terms in variable separable form,

$$\Rightarrow dx=\frac{v^2}{v^2-a^2}dv$$

Integrating

$$\int dx=\int\frac{v^2}{v^2-a^2}dv$$

$$x+c=\int\frac{v^2-a^2+a^2}{v^2-a^2}dv$$

$$x+c=\int dv+\int\frac{a^2}{v^2-a^2}dv$$

$$x+c=v+\frac{a^2}{2a}\log\left|\frac{v-a}{v+a}\right|$$

$$x+c=x-y+\frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right|$$

$$c=-y+\frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right|$$

$$y-\frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right|=C$$

**Illustration 6: Solve, $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$**

**Ans:** Let $z=x+y$ and differentiate it to get the variable separable form as

$$\therefore \frac{dz}{dx}=1+\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{dz}{dx}-1$$

$$\frac{dz}{dx}-1=\sin z+\cos z$$

$$\frac{dz}{dx}=\sin z+\cos z+1$$

Using identities $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x=2\cos^2\frac{x}{2}-1$,

$$\Rightarrow \frac{dz}{dx}=2\sin\frac{z}{2}\cos\frac{z}{2}+2\cos^2\frac{z}{2}$$

Taking out $2\cos^2\frac{z}{2}$,

$$\Rightarrow \frac{dz}{dx}=2\cos^2\frac{z}{2}\left(\tan\frac{z}{2}+1\right)$$

$$\Rightarrow \frac{dz}{2\cos^2\frac{z}{2}\left(\tan\frac{z}{2}+1\right)}=dx$$

Integrating,

$$\Rightarrow \int\frac{dz}{2\cos^2\frac{z}{2}\left(\tan\frac{z}{2}+1\right)}=\int dx$$

Take $u=\tan\frac{z}{2}+1$.

So, $du=\frac{1}{2}\sec^2\frac{z}{2}dz$

Using identities $\sec x=\frac{1}{\cos x}$,

$$du=\frac{1}{2\cos^2\frac{z}{2}}dz$$

Substituting in the integral,

$$\Rightarrow \int\frac{du}{u}=\int dx$$

$$\Rightarrow \log u=x+c$$

Resubstituting back $u=\tan\frac{z}{2}+1$,

$$\Rightarrow \log\left(\tan\frac{z}{2}+1\right)=x+c$$

$$\therefore \log\left(\tan\frac{x+y}{2}+1\right)=x+c$$

This is the required general solution.

**iii. Solution of differential equation of the type**

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2},\ \text{where}\ \frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$$

Here $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ ...(i)

Let $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\lambda$ (say)

$\therefore a_1=\lambda a_2, b_1=\lambda b_2$

From (i), $\frac{dy}{dx}=\frac{\lambda a_2x+\lambda b_2y+c_1}{a_2x+b_2y+c_2}$

$$=\frac{dy}{dx}=\frac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}\quad ...(ii)$$

Let $z=a_2x+b_2y$

$$\therefore \frac{dz}{dx}=a_2+b_2\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{\frac{dz}{dx}-a_2}{b_2}\quad ...(iii)$$

From (ii) and (iii), we get

$$\frac{\frac{dz}{dx}-a_2}{b_2}=\frac{\lambda z+c_1}{z+c_2}$$

or, $\frac{dz}{dx}=\frac{b_2\lambda z+c_1}{z+c_2}+a_2=\frac{\lambda b_2z+b_2c_1+a_2z+a_2c_2}{z+c_2}$

or $dx=\frac{z+c_2}{\lambda b_2+a_2z+b_2c_1+a_2c_2}dz$, where $x$ and $z$ are separated.

Integrating, we get

$$x+c=\int\frac{z+c_2}{\lambda b_2+a_2z+b_2c_1+a_2c_2}dz \text{ where } z=a_2x+b_2y$$

**2. Method – 2**

**i. Homogeneous Differential Equation:**

A function $f(x,y)$ is called homogeneous function of degree n if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$

For example:

a) $f(x,y)=x^2y^2-xy^3$ is a homogeneous function of degree four, since

$$f(\lambda x,\lambda y)=(\lambda^2x^2)(\lambda^2y^2)-(\lambda x)(\lambda^3y^3)$$
$$=\lambda^4(x^2y^2-xy^3)$$
$$=\lambda f(x,y)$$

b) $f(x,y)=x^2e^{\frac{x}{y}}+\frac{x^3}{y}+y^2\log\left(\frac{y}{x}\right)$ is a homogeneous function of degree two, since

$$f(\lambda x,\lambda y)=(\lambda^2x^2)e^{\frac{\lambda x}{\lambda y}}+\frac{\lambda^3x^3}{\lambda y}+(\lambda^2y^2)\log\left(\frac{\lambda y}{\lambda x}\right)$$
$$=\lambda^2\left[x^2e^{\frac{x}{y}}+\frac{x^3}{y}+y^2\log\left(\frac{y}{x}\right)\right]$$
$$=\lambda^2 f(x,y)$$

A differential equation of the form $\frac{dy}{dx}=f(x,y)$, where $f(x,y)$ is a homogeneous polynomial of degree zero is called a homogeneous differential equation. Such equations are solved by substituting $v=\frac{y}{x}$ or $\frac{x}{y}$ and then separating the variables.

**Illustration 7: Solve $\frac{dy}{dx}=\frac{y(2y-x)}{x(2y+x)}$**

**Ans:** Each of the given functions, i.e. $y(2y-x)$ and $x(2y+x)$ is a homogeneous function of degree $2$. Hence, the given equation is a homogeneous differential equation.

Putting $y=vx$ and differentiating w.r.t $x$,

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting in given equation,

$$v+x\frac{dv}{dx}=\frac{vx(2vx-x)}{x(2vx+x)}=\frac{v(2v-1)}{2v+1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v(2v-1)}{2v+1}-v$$

After simplifying the RHS,

$\Rightarrow x\frac{dv}{dx}=\frac{-2v}{2v+1}$

$\Rightarrow \frac{2v+1}{2v}dv=-\frac{1}{x}dx$

Integrating,

$\Rightarrow \int\frac{2v+1}{2v}dv=-\int\frac{1}{x}dx$

$\Rightarrow \int dv+\int\frac{1}{2v}dv=-\int\frac{1}{x}dx$

$\Rightarrow v+\frac{1}{2}\log v=-\log x+\log c$

Resubstituting $v=\frac{y}{x}$,

$\Rightarrow \frac{y}{x}+\frac{1}{2}\log\frac{y}{x}=-\log x+\log c$

$\therefore y+\frac{x}{2}\log\frac{y}{x}=-x\log x+C$

**ii. Differential Equation Reducible to Homogeneous Forms:**

Equation of the form $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$, where $\frac{a}{a'}\neq\frac{b}{b'}$ can be reduced to homogeneous form by changing the variables $x, y$ to $x', y'$ by equations $x=x'+h$ and $y=y'+k$ where $h$ and $k$ are constants to be chosen so as to make the given equation homogeneous.

$dx=dx'$ and $dy=dy'$

The given equation becomes

$\frac{dy'}{dx'}=\frac{a(x'+h)+b(y'+k)+c}{a'(x'+h)+b'(y'+k)+c'}$

$=\frac{ax'+by'+(ah+bk+c)}{a'x'+b'y'+(a'h+b'k+c')}$

Now, choose $h$ and $k$ so that

$ah+bk+c=0$

and $a'h+b'k+c'=0$

From these equation, the values of $h$ and $k$ in terms of the coefficients are obtained.

Then the given equation reduces to

$\frac{dy}{dx}=\frac{ax'+by'}{a'x'+b'y'}$

Which is the homogeneous form.

**3. Method – 3**

**i. Linear Differential Equation:**

A differential equation is said to be linear if the dependent variable y and its derivative occur in the first degree.

An equation of the form $\frac{dy}{dx}+Py=Q$ ...(i)

where $P$ and $Q$ are functions of x only or constant is called a linear equation of the first order.

Similarly $\frac{dx}{dy}+Px=Q$ is a linear differential equation where $P$ and $Q$ are functions of $y$ only.

To get the general solution of the above equations determination of a function $R$ of $x$ called Integrating function (I.F) is required. So, multiply both sides of the given equation by $R$

where, $R=e^{\int Pdx}=\text{I.F.}$ ...(iii)

From (i) and (iii),

$$e^{\int Pdx}\frac{dy}{dx}+Pye^{\int Pdx}=Qe^{\int Pdx}$$

$$\frac{d}{dx}\left(ye^{\int Pdx}\right)=Q.e^{\int Pdx}$$

Integrating,

$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$ is the required solution.

This can also be written and memorized as

$$y(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

**Illustration 8: Solve $2x\frac{dy}{dx}=y+6x^{\frac{5}{2}}-2\sqrt{x}$**

**Ans:** The given equation can be written as

$$\frac{dy}{dx}+\left(\frac{-1}{2x}\right)y=3x^{\frac{3}{2}}-\frac{1}{\sqrt{x}}$$

This is the form of $\frac{dy}{dx}+Py=Q$

Hence $\text{I.F.}=e^{\int\frac{-1}{2x}dx}=e^{-\frac{1}{2}\log x}=\frac{1}{\sqrt{x}}$

Now using $y(\text{I.F.}) = \int Q(\text{I.F.})dx + c$,

$\Rightarrow \frac{y}{\sqrt{x}} = \int\left(3x^{\frac{3}{2}} - \frac{1}{\sqrt{x}}\right)\frac{1}{\sqrt{x}}dx + c$

$\Rightarrow \frac{y}{\sqrt{x}} = \int\left(3x - \frac{1}{x}\right)dx + c$

Integrating,

$\Rightarrow \frac{y}{\sqrt{x}} = 3\frac{x^2}{2} - \log x + c$

$\Rightarrow y = \frac{3}{2}x^2\sqrt{x} - \sqrt{x}\log x + c\sqrt{x}$

Therefore, $y = \frac{3}{2}x^{\frac{5}{2}} - \sqrt{x}\log x + c\sqrt{x}$.

**ii. Differential Equation Reducible to the Linear Form:**

Sometimes equations which are not linear can be reduced to the linear form by suitable transformation.

Here, $f'(y)\frac{dy}{dx} + f(y)P(x) = Q(x)$ ...(i)

Let, $f(y) = u \Rightarrow f'(y)dy = du$

Then (i) reduces to

$\frac{du}{dx} + uP(x) = Q(x)$ Which is of the linear differential equation form.

**Illustration 9: Solve $\sec^2\theta d\theta + \tan\theta(1 - r\tan\theta)dr = 0$**

**Ans:** The given equation can be written as

$\frac{d\theta}{dr} + \frac{\tan\theta}{\sec^2\theta} = \frac{r\tan^2\theta}{\sec^2\theta}$

$\left(\frac{\sec^2\theta}{\tan^2\theta}\right)\frac{d\theta}{dr} + \frac{1}{\tan\theta} = r$

$\csc^2\theta\frac{d\theta}{dr} + \cot\theta = r$

Let $\cot\theta = u$

$\Rightarrow -\csc^2\theta d\theta = du$

Then (i) reduces to

$-\frac{du}{dr} + u = r$ or $\frac{du}{dr} - u = -r$

Which is a linear differential equation.

So, $I.F. = e^{\int -1 dr} = e^{-r}$

Now using $y(I.F.) = \int Q(I.F.)dx + c$,

$\Rightarrow ue^{-r} = \int -re^{-r}dr + c$

$\Rightarrow ue^{-r} = -\int re^{-r}dr + c$

Using integration by parts with first function as $r$ and second function as $e^{-r}$,

$\Rightarrow ue^{-r} = -\left[ r\int e^{-r}dr - \int \frac{d}{dr}(r).\int e^{-r}dr \right]$

$\Rightarrow ue^{-r} = -\left[ -re^{-r} - \int e^{-r}dr \right]$

$\Rightarrow ue^{-r} = -\left[ -re^{-r} + e^{-r} \right] + c$

$\Rightarrow ue^{-r} = re^{-r} - e^{-r} + c$

$\Rightarrow u = r - 1 + C$

Resubstituting,

$\therefore \cot\theta = r - 1 + C$

**iii. Extended Form of Linear Equations:**

**Bernoulli's Equation:**

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where $P$ and $Q$ are function of $x$ alone or constants and $n$ is constant, other than $0$ and $1$, is called a Bernoulli's equation.

Here $\frac{dy}{dx} + Py = Qy^n$

Dividing by $y^n$,

$\frac{1}{y^n}\frac{dy}{dx} + P.\frac{1}{y^{n-1}} = Q$

Putting $\frac{1}{y^{n-1}} = v$ and differentiating w.r.t $x$,

$-\frac{(n-1)}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{1}{y^n}\frac{dy}{dx} = \frac{-1}{n-1}\frac{dv}{dx}$

$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

The equation becomes

$$\frac{dv}{dx}+(1-n)Pv=Q(1-n)$$

Which is a linear equation with $v$ as independent variable.

**Illustration 10: Solve $\cos^2 x\frac{dy}{dx}-y\tan 2x=\cos^4 x$, where $|x|=\frac{\pi}{4}$ and $y\left(\frac{\pi}{4}\right)=\frac{3\sqrt{3}}{8}$.**

**Ans:** The given equation can be written as

$$\frac{dy}{dx}-y\tan 2x\sec^2 x=\cos^2 x$$

This is the form of $\frac{dy}{dx}+Py=Q$

Here $P=-\tan 2x\sec^2 x, Q=\cos^2 x$

$$\int Pdx=-\int \tan 2x\sec^2 xdx$$

$$=-\int\frac{2\tan x}{1-\tan^2 x}\sec^2 xdx$$

$$=\int\frac{dt}{t}$$

Putting $1-\tan^2 x=t$

$\therefore -2\tan x\sec^2 xdx=dt$

$=\log t=\log\left(1-\tan^2 x\right)$

$\therefore$ I.F. $=e^{\int P.dx}=e^{\log\left(1-\tan^2 x\right)}=1-\tan^2 x$

Now using $y(\text{I.F.})=\int Q(\text{I.F.})dx+c$,

$\Rightarrow y\left(1-\tan^2 x\right)=\int\cos^2 x\left(1-\tan^2 x\right)dx+c$

$\Rightarrow y\left(1-\tan^2 x\right)=\int\left(\cos^2 x-\sin^2 x\right)dx+c$

Using identity $\cos 2x=\cos^2 x-\sin^2 x$,

$\Rightarrow y\left(1-\tan^2 x\right)=\int\cos 2xdx+c$

$\Rightarrow y\left(1-\tan^2 x\right)=\frac{\sin 2x}{2}+C$ .....(i)

Given that $x=\frac{\pi}{6}, y=\frac{3\sqrt{3}}{8}$

Substituting in (i),

$$\Rightarrow \frac{3\sqrt{3}}{8}\left(1-\tan^2\frac{\pi}{6}\right)=\frac{\sin\frac{\pi}{3}}{2}+C$$

$$\Rightarrow \frac{3\sqrt{3}}{8}\left(1-\frac{1}{3}\right)=\frac{\sqrt{3}}{4}+C$$

$$\Rightarrow \frac{3\sqrt{3}}{8}\left(\frac{2}{3}\right)=\frac{\sqrt{3}}{4}+C$$

$$\Rightarrow \frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{4}+C$$

$\therefore C=0$

Hence from (i),

$$\Rightarrow y\left(1-\tan^2 x\right)=\frac{\sin 2x}{2}$$

$$\therefore y=\frac{\sin 2x}{2\left(1-\tan^2 x\right)}$$

**4. Method – 4**

**Exact Differential Equation:**

A differential equation is said to be exact if it can be derived from its solution (primitive) directly by differentiation, without any elimination, multiplication etc. For example, the differential equation $xdy+ydx=0$ is an exact differential equation as it is derived by direct differentiation for its solution, the function $xy=c$.

**Illustration 11: Solve $(1+xy)ydx+(1-xy)xdy=0$**

**Ans:** The given equation can be written as

$$ydx+xy^2dx+xdy-x^2ydy=0$$

$$(ydx+xdy)+xy(ydx-xdy)=0$$

$$d(xy)+xy(ydx-xdy)=0$$

Dividing by $x^2y^2$,

$$\frac{d(xy)}{x^2y^2}+\frac{ydx-xdy}{xy}=0$$

$$\frac{d(xy)}{x^2y^2}+\frac{dx}{x}-\frac{dy}{y}=0$$

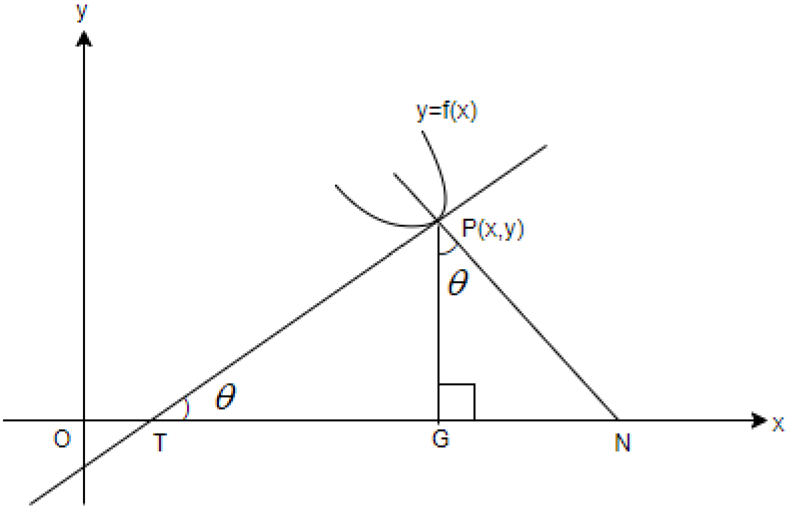
Integrating,

$$-\frac{1}{xy}+\log x-\log y=c$$

Which is the required solution.

- **Application of Differential Equations:**

The below results are helpful when solving geometrical problems.
Consider the below diagram,




Let PT and PN be the tangent and the normal at $P(x,y)$ respectively. Let the tangent at P make an angle $\theta$ with the x-axis.

Then the slope of the tangent at P $=\tan\theta=\left(\frac{dy}{dx}\right)_P$

The slope of the normal at P $=-\frac{1}{\left(\frac{dy}{dx}\right)_P}$

Equation of the tangent at $P(x,y)$ is

$$Y-y=\left(\frac{dy}{dx}\right)_P(X-x)$$

Equation of the normal at $P(x,y)$ is

$$Y-y=-\frac{1}{\left(\frac{dy}{dx}\right)_P}(X-x)$$

From $\Delta PGT, \sin\theta=\frac{PG}{PT}=\frac{y}{PT}$

$$\therefore \boxed{\begin{array}{l} PT = y\,\text{cosec}\,\theta\,(\text{length of the tangent}) \\ = y\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta} = y\dfrac{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}} \end{array}}$$

And $\tan\theta = \dfrac{PG}{TG} = \dfrac{y}{TG}$

$$\Rightarrow \boxed{TG = y\cot\theta\,(\text{length of the sub tangent}) = \dfrac{y}{\dfrac{dy}{dx}}}$$

From $\Delta PGN, \cos\theta = \dfrac{PG}{PN} = \dfrac{y}{PN}$

$$\Rightarrow \boxed{\begin{array}{l} PN = y\sec\theta\,(\text{length of the normal}) \\ = y\sqrt{1+\tan^2\theta} = y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} \end{array}}$$

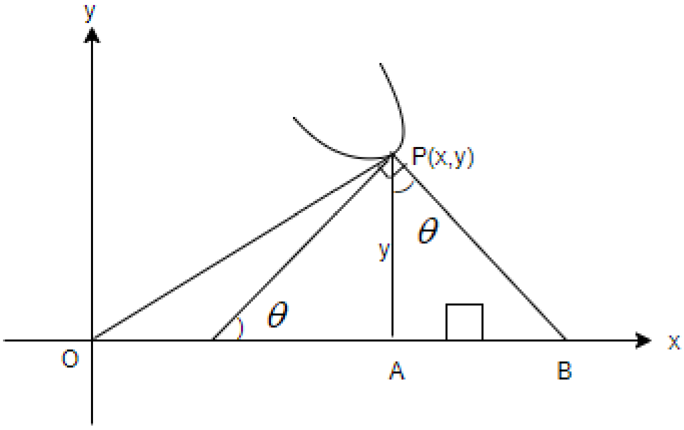
$\tan\theta = \dfrac{GN}{y}$

$$\Rightarrow \boxed{GN = y\tan\theta = y\dfrac{dy}{dx}(\text{length of the sub normal})}$$

**Illustration 12: If the length of the sub-normal at any point P on the curve is directly proportional to $OP^2$, where O is the origin, then form the differential equation of the family of curves and hence find the family of curves.**

**Ans:** Here $AB = y\tan\theta = y\dfrac{dy}{dx}$

Drawing the diagram,




Also $OP^2 = x^2 + y^2$

Given, length of the subnormal $= k.OP^2$

$$y\frac{dy}{dx} = k\left(x^2 + y^2\right)$$

$$2y\frac{dy}{dx} - 2ky^2 = 2kx^2 \quad \ldots.(i)$$

Let $y^2 = t \Rightarrow 2y\frac{dy}{dt} = \frac{dt}{dx} \quad \ldots(ii)$

From (i) and (ii),

$$\frac{dt}{dx} - 2kt = 2kx^2$$

Which is a linear differential equation.

$$\therefore \text{I.F.} = e^{\int -2kdx} = e^{-2kx}$$

The solution is

$$t.e^{-2kx} = \int 2kx^2 e^{-2kx} dx + c$$

$$= 2k\left[x^2\frac{e^{-2kx}}{-2k} + \frac{2}{2k}\int xe^{-2kx}dx\right]$$

$$= 2k\left[x^2\frac{e^{-2kx}}{-2k} + \frac{1}{k}\left\{x\frac{e^{-2kx}}{-2k} + \frac{1}{2k}\int e^{-2kx}dx\right\}\right]$$

$$= -x^2 e^{-2kx} - \frac{xe^{-2kx}}{k} + \frac{1}{k}\frac{e^{-2kx}}{2k} + c$$

$$\therefore y^2 = -x^2 - \frac{x}{k} + \frac{1}{2k^2} + ce^{2kx}$$