Revision Notes

Class 12 Maths

Chapter 8 – Definite Integration and Area

Definite Integration

1. Definition:

If $F(x)$ is an antiderivative of $f(x)$, then $F(b)–F(a)$ is known as the definite integral of $f(x)$ from a to b, such that the variable x, takes any two independent values say a and b.

This is also denoted as $\int_a^b f(x)dx$.

Thus $\int_a^b f(x)dx = F(b) - F(a)$, The numbers a and b are called the limits of integration; a is the lower limit and b is the upper limit. Usually $F(b) - F(a)$ is abbreviated by writing $F(x)\big|_a^b$.

2. Properties of Definite Integrals:

I. $\int_a^b f(x)dx = -\int_b^a f(x)$

II. $\int_a^b f(x)dx = \int_b^a f(y)dy$

III. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may or may not lie between a and b.

IV. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

V. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Note:

(a) $\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx = \frac{a}{2}$

(b) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx=\frac{b-a}{2}$

VI. $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$

$$=\begin{cases} 0 & \text{if } f(2a-x)=-f(x) \\ 2\int_0^a f(x)dx & \text{if } f(2a-x)=f(x) \end{cases}$$

VII. $\int_{-a}^{a} f(x)dx=\begin{cases} 2\int_0^a f(x)dx & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is even} \\ 0 & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

VIII. If $f(x)$ is a periodic function of period 'a', i.e. $f(a+x)=f(x)$, then

(a) $\int_0^{na} f(x)dx=n\int_0^a f(x)dx$

(b) $\int_0^{na} f(x)dx=(n-1)\int_0^a f(x)dx$

(c) $\int_{na}^{b+na} f(x)dx=\int_0^b f(x)dx$, where $b\in R$

(d) $\int_b^{b+a} f(x)dx$ independent of b.

(e) $\int_b^{b+na} f(x)dx=n\int_0^a f(x)dx$, where $n\in 1$

IX. If $f(x)\ge 0$ on the interval $[a,b]$, then $\int_a^b f(x)dx\ge 0$.

X. If $f(x)\le g(x)$ on the interval $[a,b]$, then $\int_a^b f(x)dx\le\int_a^b g(x)dx$

XI. $\left|\int_a^b f(x)dx\right|\le\int_a^b |f(x)|dx$

XII. If $f(x)$ is continuous on $[a,b]$, m is the least and M is the greatest value of $f(x)$

on $[a,b]$, then

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a)$$

XIII. For any two functions $f(x)$ and $g(x)$, integral on the interval $[a,b]$, the Schwarz-Bunyakovsky inequality holds

$$\left|\int_a^b f(x).g(x)dx\right|\le\sqrt{\int_a^b f^2(x)dx.\int_a^b g^2(x)dx}$$

XIV. If a function $f(x)$ is continuous on the interval $[a,b]$, then there exists a point $c\in(a,b)$ such that

$$\int_a^b f(x)dx=f(c)(b-a),\text{ where } a<c<b.$$

3. Differentiation under Integral sign:

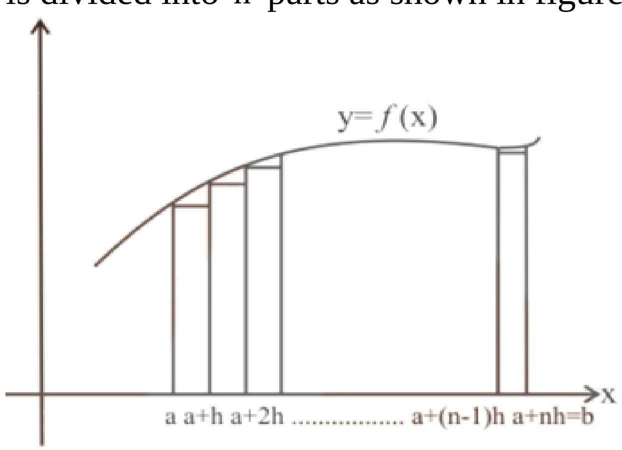
Newton Leibnitz's Theorem:

Given that x has two differential functions, $g(x)$ and $h(x)$, where $x\in[a,b]$ and f is continuous in that interval, then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right]=\frac{d}{dx}[h(x)].f[h(x)]-\frac{d}{dx}[g(x)].f[g(x)]$$

4. Definite Integral as a Limit of Sum:

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a,b]$ which is divided into n parts as shown in figure.



The point of division on x-axis are

$a, a+h, a+2h.....a+(n-1)h, a+nh,$ where $\frac{b-a}{n}=h$.

Let S_n denotes the area of these n rectangles.

Then, $S_n = hf(a)+hf(a+h)+hf(a+2h)+....+hf(a+(n-1)h)$

Clearly, S_n is area very close to the area of the region bounded by

Curve $y=f(x)$, x-axis and the ordinates $x=a$, $x=b$.

Hence $\int_a^b f(x)dx = \underset{n\to\infty}{Lt} S_n$

$$\int_a^b f(x)dx = \underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} hf(a+rh)$$

$$= \underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)r}{n}\right)$$

Note:

(a) We can also write

$S_n = hf(a+h)+hf(a+2h)+.....+hf(a+nh)$

$$\int_a^b f(x)dx = \underset{n\to\infty}{Lt} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right)r\right)$$

(b) If $a=0, b=1, \int_0^1 f(x)dx = \underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

- **Steps to express the limit of sum as definite integral**

Step 1. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\underset{n\to\infty}{Lt} \sum$ by $\int$

Step 2. Evaluate $\underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For example $\underset{n\to\infty}{Lt} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^p f(x)dx$

$$\left[\underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)\Big|_{r=1}=0,\ \underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)\Big|_{r=np}=p\right]$$

5. REDUCTION FORMULAE IN DEFINITE INTEGRALS

I. If $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$, then show that $I_n=\left(\frac{n-1}{n}\right)I_{n-2}$

Proof: $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$

$$I_n=\left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\sin^{n-2}x.\cos^2 x\,dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x.\left(1-\sin^2 x\right)dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\,dx$$

$$I_n+(n-1)I_n=(n-1)I_{n-2}$$

$$I_n=\left(\frac{n-1}{n}\right)I_{n-2}$$

Note:

(a) $\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

(b) $I_n=\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....I_0 \text{ or } I_1$ according as n is even or odd, $I_0=\frac{\pi}{2}, I_1=1$

Hence $I_n=\begin{cases}\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right).....\left(\frac{1}{2}\right).\frac{\pi}{2} & \text{if } n \text{ is even}\\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right).....\left(\frac{2}{3}\right).1 & \text{if } n \text{ is odd}\end{cases}$

II. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Proof: $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} . \tan^2 x dx$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n\ \ 2} \sec^2 x dx \quad \int_0^{\frac{\pi}{4}} (\tan x)^{n\ \ 2} dx$$

$$= \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

III. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x . \cos^n x dx$, then show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2}, n$

Proof: $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x \left(\sin x \cos^n x\right) dx$

$$= \left[-\frac{\sin^{m-1} x . \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1}}{n+1} (m-1) \sin^{m-2} x \cos x dx$$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x . \cos^n x . \cos^2 x dx$$

$$\left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}}\left(\sin^{m-2}x.\cos^n x \quad \sin^m x.\cos^n x\right)dx$$

$$=\left(\frac{m-1}{n+1}\right)I_{m-2,n}-\left(\frac{m-1}{n+1}\right)I_{m,n}$$

$$\Rightarrow\left(1+\frac{m-1}{n+1}\right)I_{m,n}=\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$I_{m,n}=\left(\frac{m-1}{m+n}\right)I_{m-2,n}$$

Note:

(a) $I_{m,n}=\left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right).....I_{0,n}\text{ or }I_{1,n}$ according as m is even or odd.

$$I_{0,n}=\int_0^{\frac{\pi}{2}}\cos^n x dx \text{ and } I_{1,n}=\int_0^{\frac{\pi}{2}}\sin x.\cos^n x dx=\frac{1}{n+1}$$

(b) Walli's Formula

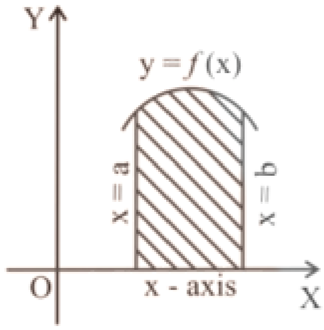
$$I_{m,n}=\begin{cases}\dfrac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5)....}{(m+n)(m+n-2)(m+n-4).....} & \text{when both m,n are even}\\[2ex] \dfrac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4)....} & \text{otherwise}\end{cases}$$

(where $b>a$) is given by

$$A=\int_a^b|y|dx=\int_a^b|f(x)|dx$$

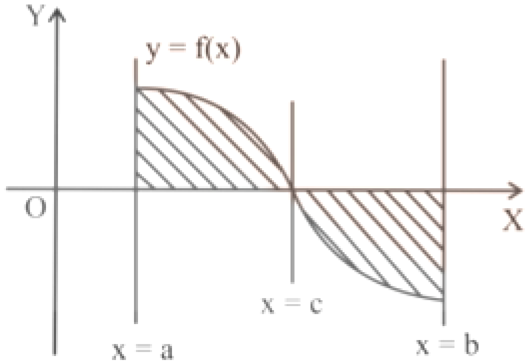
(i) If $f(x)>0\ \forall x\in[a,b]$

Then $A=\int_a^b f(x)dx$



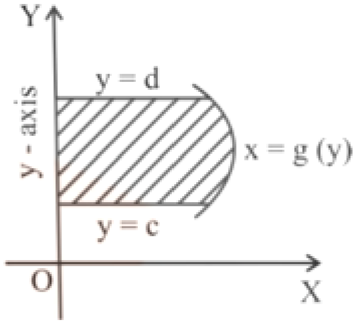
(ii) If $f(x)>0\forall x\in[a,c)\&<0\forall x\in(c,b]$

Then $A=\left|\int_a^c ydx\right|+\left|\int_c^b ydx\right|=\int_a^c f(x)dx-\int_c^b f(x)dx$ where c is a point in between a and b.



II. The area bounded by the curve $x=g(y)$, y-axis and the abscissae $y=c$ and $y=d$ (where $d>c$) is given by

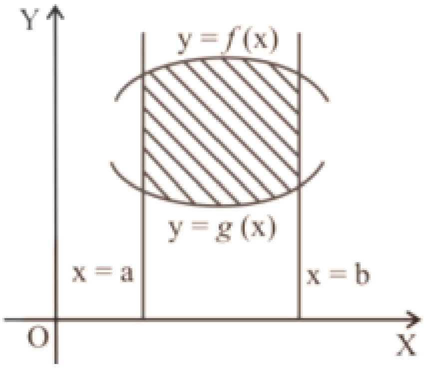
$A=\int_c^d |x|dy=\int_c^d |g(y)|dy$



III. If we have two curve $y=f(x)$ and $y=g(x)$, such that $y=f(x)$ lies above the curve $y=g(x)$ then the area bounded between them and the ordinates $x=a$ and $x=b$ $(b>a)$, is given by

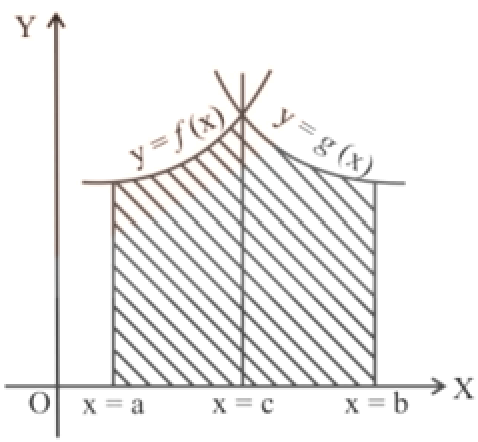
$$A=\int_a^b f(x)dx-\int_a^b g(x)dx$$

i.e. $\text{upper curve area} - \text{lower curve area}$.



IV. The area bounded by the curves $y=f(x)$ and $y=g(x)$ between the ordinates $x=a$ and $x=b$ is given by

$A=\int_a^c f(x)dx+\int_c^b g(x)dx$, where $x=c$ is the point of intersection of the two curves.



V. Curve Tracing

It is necessary to have a rough sketch of the desired piece in order to locate the area enclosed by many curves. The following steps are very useful in tracing a cartesian curve $f(x,y)=0$.

Step 1: Symmetry

(i) If all the powers of y in the equation of the given curve are even then the curve is symmetrical about x-axis.

(ii) If all the powers of x in the equation of the given curve are even then the curve is symmetrical about y-axis.

(iii) If the equation of the given curve remains unchanged on interchanging x and y, then the curve is symmetrical about the line $y = x$.

(iv) If the equation of the given curve remains unchanged when x and y are replaced by $-x$ and $-y$ respectively, then the curve is symmetrical in opposite quadrants.

Step 2: Origin

If the algebraic curve's equation contains no constant term, the curve passes through the origin.

The tangents at the origin are then calculated by equating the lowest degree terms in the equation of the specified algebraic curve to zero.

For example, the curve $y^3 = x^3 + axy$ passes through the origin and the tangents at the origin are given by $axy = 0$ i.e. $x = 0$ and $y = 0$.

Step 3: Intersection with the Co-ordinates Axes

(i) Find out the corresponding values of x by putting $y = 0$ in the equation of the given curve, to estimate the points of intersection of the curve with x-axis

(ii) Find out the corresponding values of y by putting $x = 0$ in the equation of the given curve, to estimate the points of intersection of the curve with y-axis

Step 4: Asymptotes

Find out the asymptotes of the curve.

(i) The vertical asymptotes of the given algebraic curve, or asymptotes parallel to the y-axis, are derived by equating the coefficient of the highest power of y in the equation of the supplied curve to zero.

(ii) The horizontal asymptotes of the given algebraic curve, or asymptotes parallel to the x-axis, are derived by equating the coefficient of the highest power of x in the equation of the supplied curve to zero.

Step 5: Region

Find out the regions of the plane in which no part of the curve lies. To determine such regions we solve the given equation for y in terms of x or vice-versa. Suppose that y becomes imaginary for $x > a$, the curve does not lie in the region $x > a$.

Step 6: Critical Points

Find out the values of x at which $\frac{dy}{dx} = 0$

At such points y generally changes its character from an increasing function of x to a decreasing function of x or vice-versa.

Step 7: Trace the curve with the help of the above points.

SOLVED EXAMPLES

DEFINITE INTEGRATION

Example-1

Evaluate the following integrals:

(i) $\int_2^3 x^2 dx$

(ii) $\int_1^3 \frac{x}{(x+1)(x+2)} dx$

Ans:

(i) $\int_2^3 x^2 dx$

$= \left[\frac{x^3}{3}\right]_2^3$

$= \frac{27}{3} - \frac{8}{3}$

$= \frac{19}{3}$

(ii) $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$ [Partial Fractions]

$$\int_1^3 \frac{x}{(x+1)(x+2)}dx$$
$$=\left[-\log|x+1|+2\log|x+2|\right]_1^3$$
$$=\left[-\log|4|+2\log|5|\right]-\left[-\log|2|+2\log|3|\right]$$
$$=[-\log 4+2\log 5]-[-\log 2+2\log 3]$$
$$=-2\log 2+2\log 5+\log 2-2\log 3$$
$$=-\log 2+\log 25-\log 9=\log 25-\log 18$$
$$=\log\frac{25}{18}$$

Example-2

Evaluate: $\int_0^{\frac{\pi}{4}} \sec x.\sqrt{\frac{1-\sin x}{1+\sin x}}dx$

Ans:

$$I=\int_0^{\frac{\pi}{4}} \sec x.\sqrt{\frac{1-\sin x}{1+\sin x}}dx$$
$$=\int_0^{\frac{\pi}{4}} \sec x.\sqrt{\frac{1-\sin x}{1+\sin x}}.\sqrt{\frac{1-\sin x}{1+\sin x}}dx$$
$$=\int_0^{\frac{\pi}{4}} \sec x\frac{1-\sin x}{\sqrt{1-\sin^2 x}}dx$$
$$=\int_0^{\frac{\pi}{4}} \sec x\frac{1-\sin x}{\cos x}dx$$
$$=\int_0^{\frac{\pi}{4}} \left(\sec^2 x-\sec x\tan x\right)dx$$
$$=\int_0^{\frac{\pi}{4}} \sec^2 x dx-\int_0^{\frac{\pi}{4}} \sec x\tan x dx$$

$$=[\tan x]_0^{\frac{\pi}{4}}-[\sec x]_0^{\frac{\pi}{4}}$$

$$=\left(\tan\frac{\pi}{4}-\tan 0\right)-\left(\sec\frac{\pi}{4}-\sec 0\right)$$

$$=(1-0)-\left(\sqrt{2}-1\right)=2-\sqrt{2}.$$

Example-3

Evaluate: $\displaystyle\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$

Ans:

Let $I=\displaystyle\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$

Put $x^5=t$ so that $5x^4dx=dt$.
When $x=-1, t=-1$. When $x=1, t=1$.

$$I=\int_{-1}^{1}\sqrt{t+1}\,dt$$

$$=\left[\frac{(t+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{1}=\frac{2}{3}\left[(t+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$=\frac{2}{3}\left[2^{\frac{3}{2}}-0\right]=\frac{4\sqrt{2}}{3}.$$

Example-4

Prove that $\displaystyle\int_{0}^{\frac{\pi}{2}}\sqrt{\sin\phi}\cos^5\phi\,d\phi=\frac{64}{231}$

Ans:

$$I=\int_{0}^{\frac{\pi}{2}}\sqrt{\sin\phi}\cos^5\phi\,d\phi$$

$$=\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi$$

$$=\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)^2\cos\phi\, d\phi$$

Put $\sin\phi = t$ so that $\cos\phi\, d\phi = dt$.

When $\phi = 0, \sin 0 = t \Rightarrow t=0$

When $\phi = \frac{\pi}{2}, \sin\frac{\pi}{2} = t \Rightarrow t = 1$

$$I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt = \int_0^1 \sqrt{t}\left(-2t^2+t^4\right)dt$$

$$=\int_0^1 \left(t^{\frac{1}{2}} - 2t^{\frac{5}{2}} + t^{\frac{9}{2}}\right)dt$$

$$=\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2\frac{t^{\frac{7}{2}}}{\frac{7}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}}\right]_0^1$$

$$=\left[\frac{2}{3}t^{\frac{3}{2}} - \frac{4}{7}t^{\frac{7}{2}} + \frac{2}{11}t^{\frac{11}{2}}\right]_0^1$$

$$=\left[\frac{2}{3}(1) - \frac{4}{7}(1) + \frac{2}{11}(1)\right] - [0-0+0]$$

$$=\frac{2}{3} - \frac{4}{7} + \frac{2}{11}$$

$$=\frac{154-132+42}{231} = \frac{64}{231}$$

Example-5

Evaluate: $\int_1^2 \left(\frac{x-1}{x^2}\right)e^x dx$ **or** $\int_1^2 e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

Ans:

$$\int\left(\frac{x-1}{x^2}\right)e^x dx=\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$
$$=\int\frac{1}{x}.e^x dx-\int\frac{1}{x^2}.e^x dx$$
$$=\frac{1}{x}.e^x-\int\left(-\frac{1}{x^2}\right)e^x dx-\int\frac{1}{x^2}.e^x dx \qquad \text{[Integrating first integral by parts]}$$
$$=\frac{1}{x}.e^x=F(x)$$
$$\int_1^2\left(\frac{x-1}{x^2}\right)e^x dx=\left[\frac{e^x}{x}\right]_1^2$$
$$=\frac{1}{2}.e^2-\frac{1}{1}e^1=\frac{1}{2}e^2-e.$$

AREA UNDER THE CURVES

Example-1

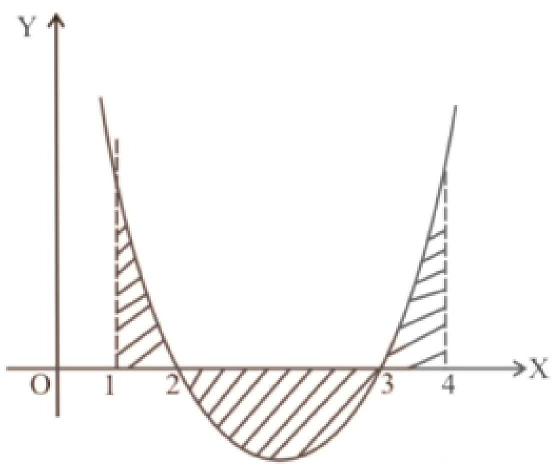
Find the area bounded by the curve $y=x^2-5x+6$, X-axis and the lines $x=1$ and 4.

Ans:

For $y=0$, we get $x^2+5x+6=0\Rightarrow x=2,3$

Hence the curve crosses X-axis at $x=2,3$ in the interval $[1,4]$.

$$\text{Bounded Area}=\left|\int_1^2 ydx\right|+\left|\int_2^3 ydx\right|+\left|\int_3^4 ydx\right|$$



$$\Rightarrow A=\left|\int_1^2 \left(x^2-5x+6\right)dx\right|+\left|\int_2^3 \left(x^2-5x+6\right)dx\right|+\left|\int_3^4 \left(x^2-5x+6\right)dx\right|$$

$$A_1=\left[\frac{2^3-1^3}{3}\right]-5\left(\frac{2^2-1^2}{2}\right)+\left[6\left(2-1\right)\right]=\frac{5}{6}$$

$$A_2=\frac{3^3-2^3}{3}-5\left(\frac{3^2-2^2}{2}\right)+6\left(3-2\right)=-\frac{1}{6}$$

$$A_3=\frac{4^3-3^3}{3}-5\left(\frac{4^2-3^2}{2}\right)+6\left(4-3\right)=\frac{5}{6}$$

$$\Rightarrow A=\frac{5}{6}+\left|-\frac{1}{6}\right|+\frac{5}{6}=\frac{11}{6}\text{sq.units}$$

Example-2

Find the area bounded by the curve: $y=\sqrt{4-x}$, X-axis and Y-axis.

Ans:

Trace the curve $y=\sqrt{4-x}$.

1. Put $y=0$ in the given curve to get $x=4$ as the point of intersection with X-axis.

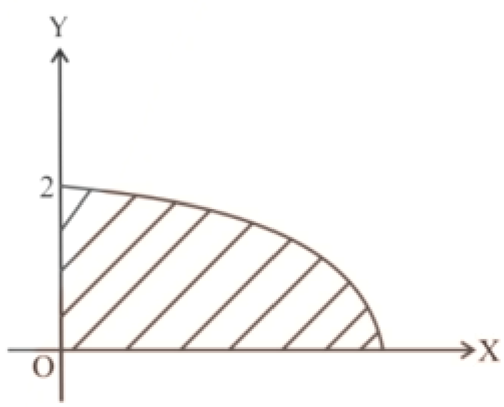
Put $x=0$ in the given curve to get $y=2$ as the point of intersection with Y-axis.

2. For the curve, $y=\sqrt{4-x}, 4-x\geq 0$

$\Rightarrow x\leq 4$

$\Rightarrow$ curve lies only to the left of $x=4$ line.

3. As any y is positive, curve is above X-axis.



Using step 1 to 3, we can draw the rough sketch of $y=\sqrt{4-x}$

In figure, Bounded area $=\int_0^4 \sqrt{4-x}\,dx=\left|\frac{-2}{3}(4-x)\sqrt{4-x}\right|_0^4=\frac{16}{3}$ sq.units

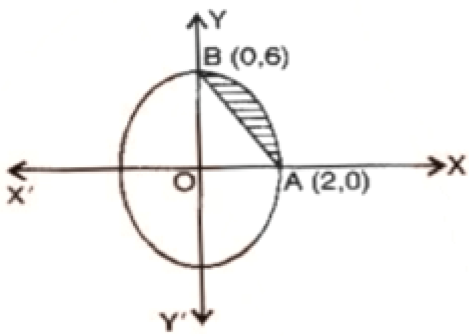
Example-3

AOBA is the part of the ellipse $9x^2+y^2=36$ in the first quadrant such that $OA=2$ and $OB=6$. Find the area between the arc AB and the chord AB.

Ans:

The given equation of the ellipse can be written as

$$\frac{x^2}{4}+\frac{y^2}{36}=1 \text{ i.e. } \frac{x^2}{2^2}+\frac{y^2}{6^2}=1$$



A is $(2,0)$ and B is $(0,6)$.

The equation of chord AB is:

$$y-0=\frac{6-0}{0-2}(x-2)$$

$\Rightarrow y=-3x+6$.

Reqd. area (shown shaded)

$$=\int_0^2 3\sqrt{4-x^2}\,dx-\int_0^2 (6-3x)\,dx$$

$$=3\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2-\left[6x-\frac{3x^2}{2}\right]_0^2$$

$$=3\left[\frac{2}{2}(0)+2\sin^{-1}(1)\right]-\left[6(2)-\frac{3(4)}{2}\right]$$

$$=3\left[2\times\frac{\pi}{2}\right]-[12-6]$$

$$=(3\pi-6)\text{sq.units}$$

Example-4

Find the area bounded by the curves $y=x^2$ and $x^2+y^2=2$ above X-axis.

Ans:

Let us first find the points of intersection of curves.

Solving $y=x^2$ and $x^2+y^2=2$ simultaneously, we get:

$$x^2+x^4=2$$

$$\Rightarrow(x^2-1)(x^2+2)=0$$

$$\Rightarrow x^2=1 \text{ and } x^2=-2 \text{ [reject]}$$

$$\Rightarrow x=\pm1$$

$$\Rightarrow A=(-1,1) \text{ and } B=(1,1)$$

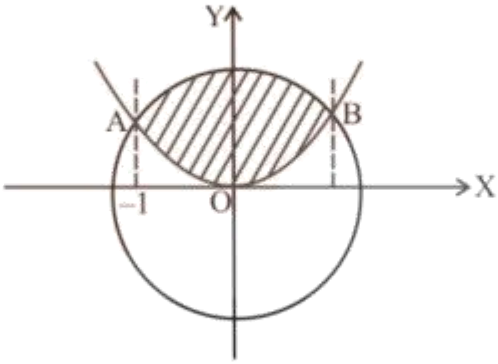
$$\text{Shaded Area}=\int_{-1}^{+1}\left(\sqrt{2-x^2}-x^2\right)dx$$

$$=\int_{-1}^{+1}\sqrt{2-x^2}dx-\int_{-1}^{+1}x^2dx$$

$$=2\int_{0}^{1}\sqrt{2-x^2}dx-2\int_{0}^{1}x^2dx$$

$$=2\left[\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1-2\left(\frac{1}{3}\right)$$

$$=2\left(\frac{1}{2}+\frac{\pi}{4}\right)-\frac{2}{3}=\frac{1}{3}+\frac{\pi}{2}\text{ sq. units}$$


Y
X
A
B
O
–1