Revision Notes

Class – 12 Mathematics

Chapter 7 - Indefinite Integration

Differentiation is the inverse of integration. Integration is the process of determining a function whose differential coefficient is known.

So from the above, if the differential coefficient of $\mathrm{F(x)}$ is $f(\mathrm{x})$

i.e. $\frac{\mathrm{d}}{\mathrm{dx}}[\mathrm{F(x)}] = f(\mathrm{x})$, then one can say that the antiderivative or integral of $f(x)$ is $F(x)$, written as $\int f(x)dx = F(x)$,

Here $\int \mathrm{dx}$ is the notation of integration $f(\mathrm{x})$ is the integrand, x is the variable of integration and dx denotes the integration with respect to X.

1. Indefinite Integral:

We know that if $\frac{\mathrm{d}}{\mathrm{dx}}[\mathrm{F(x)}] = f(\mathrm{x})$, then $\int f(\mathrm{x})\mathrm{dx} = \mathrm{F(x)}$.

Also, for any arbitrary constant C,

$$\frac{\mathrm{d}}{\mathrm{dx}}[\mathrm{F(x)} + \mathrm{C}] = \frac{\mathrm{d}}{\mathrm{dx}}[\mathrm{F(x)}] + 0 = f(\mathrm{x})$$

$$\therefore \int f(\mathrm{x})\mathrm{dx} = \mathrm{F(x)} + \mathrm{C}$$

This shows that $\mathrm{F(x)}$ and $\mathrm{F(x)+C}$ are both integrals of the same function $\mathrm{f(x)}$. Thus, for different values of C, we obtain different integrals of $\mathrm{f(x)}$. This implies that the integral of $\mathrm{f(x)}$ is not definite. By virtue of this property $\mathrm{F(x)}$ is called the indefinite integral of $\mathrm{f(x)}$.

1.1 Properties of Indefinite Integration

1. $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$

2. $\int f'(x)dx = \int \frac{d}{dx}[f(x)]dx = f(x)+c$

3. $\int k f(x)dx = k\int f(x)dx$, where k is any constant

4. If $f_1(x), f_2(x), f_3(x), \ldots$ (finite in number) are functions of x, then

$$\int [f_1(x) \pm f_2(x) \pm f_3(x) \ldots]dx$$
$$= \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \quad \ldots$$

5. If $\int f(x)dx = F(x)+c$ then $\int f(ax \pm b)dx = \frac{1}{a}F(ax \pm b)+c$

1.2 Standard Formula of Integration

The definition of an integral has the following conclusions as a direct result.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$.

2. $\int \frac{1}{x}dx = \log|x| + C$

3. $\int e^x dx = e^x + C$

4. $\int a^x dx = \frac{a^x}{\log_e a} + C$.

5. $\int \sin x dx = -\cos x + C$

6. $\int \cos x dx = \sin x + C$

7. $\int \sec^2 x dx = \tan x + C$

8. $\int \operatorname{cosec}^2 x dx = -\cot x + C$

9. $\int \sec x \tan x dx = \sec x + C$

10. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$.

11. $\int \tan x dx = -\log|\cos x| + C = \log|\sec x| + C$.

12. $\int \cot x dx = \log|\sin x| + C$

13. $\int \sec x dx = \log|\sec x + \tan x| + C$

14. $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C; |x| < 1$

16. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

17. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C; |x| > 1$

2. Methods of Integration

2.1 Method of Substitution

By using the suitable substitution, the variable x in $\int f(x)dx$ is changed into another variable t so that the integrand $f(x)$ is changed into $F(t)$ is an algebraic sum of standard integrals or a standard integral. There is no universal formula for determining a suitable substitute, and experience is the greatest guidance in this regard.

The following ideas, on the other hand, will be beneficial.

(i) If the integrand is of the form $f'(ax+b)$, then we

put $ax+b=t$ and $dx=\frac{1}{a}dt$

Thus, $\int f'(ax+b)dx=\int f'(t)\frac{dt}{a}$

$$=\frac{1}{a}\int f'(t)dt=\frac{f(t)}{a}=\frac{f(ax+b)}{a}+c$$

(ii) When the integrand is of the form $x^{n-1}f'\left(x^{n}\right)$, we put $x^{n}=t$ and $nx^{n-1}dx=dt$

Thus, $\int x^{n-1}f'\left(x^{n}\right)dx=\int f'(t)\frac{dt}{n}=\frac{1}{n}$

$$\int f'(\mathrm{t})\mathrm{dt}=\frac{1}{\mathrm{n}}f(\mathrm{t})=\frac{1}{\mathrm{n}}f\left(\mathrm{x}^{\mathrm{n}}\right)+\mathrm{c}$$

(iii) When the integrand is of the form $[f(x)]^{n}\cdot f'(x)$, we put $f(x)=t$ and $f'(x)dx=dt$

Thus,

(iv) When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put

$f(\mathrm{x})=\mathrm{t}$ and $f'(\mathrm{x})\mathrm{dx}=\mathrm{dt}$

Thus, $\int\frac{f'(\mathrm{x})}{f(\mathrm{x})}\mathrm{dx}=\int\frac{\mathrm{dt}}{\mathrm{t}}=\log|\mathrm{t}|=\log|f(\mathrm{x})|+\mathrm{c}$

2.1.1 Some Special Integrals

1. $\int\frac{dx}{x^{2}+a^{2}}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

2. $\int\frac{\mathrm{dx}}{\mathrm{x}^{2}-\mathrm{a}^{2}}=\frac{1}{2\mathrm{a}}\log\left|\frac{\mathrm{x}-\mathrm{a}}{\mathrm{x}+\mathrm{a}}\right|+\mathrm{C}$

3. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

5. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

6. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

7. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$.

8. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

9. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

2.1.2 Integrals of the Form

(a) $\int f\left(a^2 - x^2\right) dx$,

(b) $\int f\left(a^2 + x^2\right) dx$,

(c) $\int f\left(x^2 - a^2\right) dx$,

(d) $\int f\left(\frac{a-x}{a+x}\right) dx$,

Working Rule

Integral Substitution

$\int f\left(a^2 - x^2\right) dx, \quad x = a\sin\theta$ or $x = a\cos\theta$

$\int f\left(a^2+x^2\right)dx, \quad x=a\tan\theta$ or $x=a\cot\theta$

$\int f\left(x^2-a^2\right)dx, \quad x=a\sec\theta$ or $x=a\operatorname{cosec}\theta$

$\int f\left(\frac{a-x}{a+x}\right)dx$ or $\int f\left(\frac{a+x}{a-x}\right)dx \quad x=a\cos 2\theta$

2.1.3 Integrals of the Form

(a) $\int \frac{dx}{ax^2+bx+c}$

(b) $\int \frac{\mathrm{dx}}{\sqrt{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}}$

(c) $\int \sqrt{ax^2+bx+c}\,dxa$,

Working Rule

(i) Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic.

(ii) Complete the square in the terms involving x, i.e. write ax^2+bx+c in the form

$$a\left[\left(x+\frac{b}{2a}\right)^2\right]-\frac{\left(b^2-4ac\right)}{4a}$$

(iii) One of the nine special integrals is used to transform the integrand.

(iv) Then just integrate the function.

2.1.4 Integrals of the Form

(a) $\int \frac{px+q}{ax^2+bx+c}dx$

(b) $\int \frac{\mathrm{px}+\mathrm{q}}{\sqrt{ax^2+bx+c}}\mathrm{dx}$

(c) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Integral Working Rule

$\int \frac{px+q}{ax^2+bx+c}dx$ Put $px+q=\lambda(2ax+b)+\mu$ or $px+q=\lambda$ (derivative of quadratic) $+\mu$.

When the coefficient of x and the constant term on both sides are compared, we obtain

$$p=2a\lambda \text{ and } q=b\lambda+\mu \Rightarrow \lambda=\frac{}{2a} \text{ and } \mu=\left(q-\frac{bp}{2a}\right).$$

Then integral becomes

$$\int \frac{px+q}{ax^2+bx+c}dx$$

$$=\frac{2ax+b}{2a}\quad \frac{dx}{ax^2+bx+c}dx+\left(q-\frac{bp}{2a}\right)\quad \frac{d}{ax^2+bx+c}$$

$$=\frac{dx}{2a}\log\left|ax^2+bx+c\right|+\left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$$

In this case the integral becomes

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx=$$

$$\frac{p}{2a}\quad \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx+\left(q-\frac{bp}{2a}\right)\quad \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$=\frac{p}{a}\sqrt{ax^2+bx+c}+\left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx$$

In this example, the integral is transformed to

$$(px+q)\sqrt{ax^2+bx+c}dx=\frac{p}{2a}\quad(2ax+b)\sqrt{ax^2+bx+c}dx$$

$$+\left(q-\frac{bp}{2a}\right)\int\sqrt{ax^2+bx+c}dx$$

$$=\frac{p}{3a}\left(ax^2+bx+c\right)^{3/2}+\left(q-\frac{bp}{2a}\right)\int\sqrt{ax^2+bx+c}dx$$

2.1.5 Integrals of the Form

$\int\frac{P(x)}{\sqrt{ax^2+bx+c}}dx$, where $P(x)$ is a polynomial in x of degree $n\geqslant 2$.

Working Rule: Write

$$\int\frac{P(x)}{\sqrt{ax^2+bx+c}}dx=$$

$$=\left(a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}\right)\sqrt{ax^2+bx+c}+k\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

where $k, a_0, a_1, \ldots a_{n-1}$ are constants that may be found by separating the above relation and equating the coefficients of various powers of x on both sides.

2.1.6 Integrals of the Form

$$\int\frac{x^2+1}{x^4+kx^2+1}dx \quad \text{or} \quad \int\frac{x^2-1}{x^4+kx^2+1}dx$$

where k is a constant positive, negative or zero.

Working Rule

(i) x^2 is divided into the numerator and denominator.

(ii) Put $x-\frac{1}{x}=z$ or $x+\frac{1}{x}=z$, Whatever substitution yields the numerator of the resultant integrand on differentiation

(iii) In Z, evaluate the resultant integral.

(iv) In terms of X, express the outcome.

2.1.7 Integrals of the Form

$\int \frac{dx}{P\sqrt{Q}}$, where P, Q are linear or quadratic functions of x.

Integral Substitution

$$\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$$

$$cx+d=z^2$$

$$\int \frac{dx}{\left(ax^2+bx+c\right)\sqrt{px+q}} \quad px+q=z^2$$

$$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}} \quad px+q=\frac{1}{z}$$

$$\int \frac{dx}{\left(ax^2+b\right)\sqrt{cx^2+d}} \quad x=\frac{1}{z}$$

3. Method of Partial Fractions For Rational Functions

By resolving the integrand into partial fractions, Integrals of the type $\int \frac{p(x)}{g(x)}$ may be integrated.

The following is how we proceed:

First of all check the degree of $p(x)$ and $g(x)$.

If degree of $p(\mathrm{x})$ degree of $g(\mathrm{x})$, then divide $p(\mathrm{x})$ by $g(\mathrm{x})$ till its degree is less, i.e. put in the form $\frac{p(\mathrm{x})}{g(\mathrm{x})}=\mathrm{r}(\mathrm{x})+\frac{f(\mathrm{x})}{g(\mathrm{x})}$ where degree of $f(\mathrm{x})<$ degree of $g(\mathrm{x})$

CASE 1: When there are non-repeated linear components in the denominator. i.e.

$$g(\mathrm{x})=(\mathrm{x}-\alpha_1)(\mathrm{x}-\alpha_2)\ldots(\mathrm{x}-\alpha_\mathrm{n})$$

In such a case write $f(\mathrm{x})$ and g as:

$$\frac{f(\mathrm{x})}{g(\mathrm{x})}=\frac{\mathrm{A}_1}{(\mathrm{x}-\alpha_1)}+\frac{\mathrm{A}_2}{(\mathrm{x}-\alpha_2)}+\ldots+\frac{\mathrm{A}_\mathrm{n}}{(\mathrm{x}-\alpha_\mathrm{n})}$$

After extracting L.C.M., compare the coefficients of various powers on both sides to get the constants $A_1, A_2, \ldots A_n$.

CASE 2: When there are both repeated and non-repeated linear factors in the denominator.

i.e. $g(\mathrm{x})=(\mathrm{x}-\alpha_1)^2(\mathrm{x}-\alpha_3)\ldots(\mathrm{x}-\alpha_\mathrm{n})$

In such a case write $f(x)$ and $g(x)$ as:

$$\frac{f(\mathrm{x})}{g(\mathrm{x})}=\frac{\mathrm{A}_1}{\mathrm{x}-\alpha_1}+\frac{\mathrm{A}_2}{(\mathrm{x}-\alpha_1)^2}+\frac{\mathrm{A}_3}{\mathrm{x}-\alpha_3}+\ldots+\frac{\mathrm{A}_\mathrm{n}}{(\mathrm{x}-\alpha_\mathrm{n})}$$

After extracting L.C.M., compare the coefficients of various powers on both sides to get the constants $A_1, A_2, \ldots A_n$.

Note: corresponding to a linear factor that has been repeated $(\mathrm{x}-\mathrm{a})^\mathrm{r}$ in the denominator, a sum of r partial fractions of the type $\frac{\mathrm{A}_1}{\mathrm{x}-\mathrm{a}}+\frac{\mathrm{A}_2}{(\mathrm{x}-\mathrm{a})^2}+\ldots+\frac{\mathrm{A}_\mathrm{r}}{(\mathrm{x}-\mathrm{a})^r}$ is taken.

CASE 3: When there is a non-repeated quadratic component in the denominator that cannot be factored further:

$$g(x)=\left(ax^2+bx+c\right)(x-\alpha_3)(x-\alpha_4)\ldots(x-\alpha_n)$$

In such a case express $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)}=\frac{A_1x+A_2}{ax^2+bx+c}+\frac{A_3}{x-\alpha_3}+\ldots+\frac{A_n}{x-\alpha_n}$$

After extracting L.C.M., compare the coefficients of various powers on both sides to get the constants $A_1, A_2, \ldots A_n$.

CASE 4: When there is a repeating quadratic component in the denominator that cannot be factored further:

i.e. $g(x)=\left(ax^2+bx+c\right)^2(x-\alpha_5)(x-\alpha_6)\ldots(x-\alpha_n)$

In such a case write $f(x)$ and $g(x)$ as

$$\frac{f(x)}{g(x)}=\frac{A_1x+A_2}{ax^2+bx+c}+\frac{A_3x+A_4}{\left(ax^2+bx+c\right)^2}+\frac{A_5}{x-\alpha_5}+\ldots+\frac{A_n}{(x-\alpha_n)}$$

After extracting L.C.M., compare the coefficients of various powers on both sides to get the constants $A_1, A_2, \ldots A_n$.

CASE 5: If x will be the even power then

(i) Put $x^2=z$ in the integrand.

(ii) Resolve the resulting rational expression in z into partial fractions

(iii) Put $z=x^2$ again in the partial fractions and then integrate both sides.

4. Method of Integration by Parts

Integration by parts refers to the process of combining the output of two functions.

e.g. if u and v are two functions of x, then $\int (uv)\mathrm{dx} = u \cdot \int v\,\mathrm{dx} - \int\left(\frac{\mathrm{d}u}{\mathrm{dx}} \cdot \int v\mathrm{dx}\right)\mathrm{dx}$

To put it another way, the integral of the product of two functions = first function $\times$ integral of the second - integral of (differential of first $\times$ integral of the second function).

Working Hints

(i) Choose the first and second functions in such a way that the derivative and integral of the first function may be determined quickly.

(ii) In case of integrals of the form $\int f(\mathrm{x}) \cdot \mathrm{x}^{\mathrm{n}}\mathrm{dx}$, take x^{n} as the first function and $f(\mathrm{x})$ as the second function.

(iii) In case of integrals of the form $\int (\log x)^{\mathrm{n}} \cdot 1\mathrm{dx}$, take 1 as the second function and $(\log x)^n$ as the first function.

(iv) The rule of parts integration can be applied several times if necessary.

(v) If the two functions are of different types, the first function can be chosen as the the one with the first initial in the word "ILATE," where

I - Inverse Trigonometric function

L - Logarithmic function

A - Algebraic function $\mathrm{T}-$ Trigonometric function

$\mathrm{E}-$ Exponential function.

(vi) If both functions are trigonometric, use the second function as the second function with a simple integral. If both functions are algebraic, choose the one with the simpler derivative as the initial function.

(vii) If the integral is an inverse trigonometric function of an algebraic expression in x, simplify the integrand first using an appropriate trigonometric substitution before integrating the new integrand.

4.1 Integrals of the Form $\int e^{x}\left[f(x)+f'(x)\right]dx$

Working Rule

(i) Divide the integral into two equal halves.

(ii) Only use parts to integrate the first integral, i.e.

$$=\left[f(x)\cdot e^{x}-\int f'(x)\cdot e^{x}x\right]+\int e^{x}f'(x)dx$$

$$=e^{x}f(x)+C$$

4.2 Integrals of the Form:

After integrating by parts, the initial integrand reappears.

Working Rule

(i) Use the part-by-part integration technique twice.

(ii) If we integrate by parts a second time, we'll get the same integrand as before, therefore we'll set it equal to I.

(iii) On one side, transpose and gather words involving I, and assess I.

5. Integral of the Form (Trigonometric Formats)

5.1

(a) $\int \frac{dx}{a+b\cos x}$

(b) $\int \frac{dx}{a+b\sin x}$

(c) $\int \frac{dx}{a+b\cos x+c\sin x}$

Working Rule

(i) Put $\cos x=\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x=\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ so that the given integrand becomes a function of $\tan\frac{x}{2}$.

(ii) Put $\tan\frac{x}{2}=z\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}dx=dz$

(iii) Integrate the resulting rational algebraic function of z

(iv) In the answer, put $z=\tan\frac{x}{2}$.

5.2 Integrals of the Form

(a) $\int\frac{dx}{a+b\cos^2 x}$

(b) $\int\frac{dx}{a+b\sin^2 x}$

(c) $\int\frac{dx}{a\cos^2 x+b\sin x\cos x+c\sin^2 x}$

Working Rule

(i) Divide the numerator and denominator by $\cos^2 x$.

(ii) In the denominator, replace sec'x, if any, by $1+\tan^2 x$.

(iii) Put $\tan x=z\Rightarrow\sec^2 xdx=dz$

(iv) Integrate the resulting rational algebraic function of z.

(v) In the answer, put $z=\tan x$.

5.3 Integrals of the Form

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx$$

Working Role

(i) Put Numerator $=\lambda$ (denominator) $+\mu$ (derivative of denominator) $a\cos x + b\sin x = \lambda(c\cos x + d\sin x) + \mu(-c\sin x + d\cos x)$

(ii) Equate coefficients of $\sin x$ and $\cos x$ on both sides and find the values of λ, and μ.

(iii) Divide the provided integral into two parts and evaluate each independently, as follows:

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx =$$

$$\lambda\iint \mathrm{ldx} + \mu\int \frac{-\mathrm{csinx} + \mathrm{d}\cos\mathrm{x}}{\mathrm{a}\cos\mathrm{x} + \mathrm{b}\sin\mathrm{x}}\mathrm{dx} = \lambda\mathrm{x} + \mu\log|\mathrm{a}\cos\mathrm{x} + \mathrm{b}\sin\mathrm{x}|$$

(iv) Substitute the values of λ and μ found in step 2 .

5.4 Integrals of the Form $\int \frac{a + b\cos x + \mathrm{c}\sin x}{e + f\cos x + g\sin x}dx$

Working Rule

(i) Put Numerator $= I$ (denominator) $+\mathrm{m}$ (derivative of denominator) $+\mathrm{n}$

$$a + b\cos x + c\sin x = l(e + f\cos x + g\sin x) + m$$

$$(-f\sin x + g\cos x) + n$$

(ii) Equate coefficients of $\sin x, \cos x$ and constant term n on both sides and find the values of $l, \mathrm{m}, \mathrm{n}$.

(iii) The next step is to divide the provided integral into 3 distinct integrals and evaluate each one separately, i.e.

$$\int \frac{a+b\cos x+c\sin x}{e+f\cos x+g\sin x}dx$$

$$l\int 1dx \quad m\int \frac{-f\sin x+g\cos x}{e \quad f\cos x \quad g\sin x}dx$$

$$n\int \frac{dx}{e+f\cos x+g\sin x}$$

$$=Lx+m\log|e+f\cos x+g\sin x|+n\int \frac{dx}{e+f\cos x+g\sin x}dx$$

(iv) Substitute the values of l, m, n found in Step (ii). 0 |c|

5.5 Integrals of the Form $\int \sin^m x\cos^n xdx$

Working Rule

(i) If the power of $\sin x$ is an odd positive integer, put $\cos x = t$.

(ii) If the power of $\cos x$ is an odd positive integer, put $\sin x = t$.

(iii) If the power of $\sin x$ and $\cos x$ are both odd positive integers, put $\sin x = t$ or $\cos x = t$

(iv) If the power of $\sin x$ and $\cos x$ are both even positive integers, use De' Moivre's theorem as follows:

Let, $\cos x + i\sin x = z$. Then $\cos x - i\sin x = z^{-1}$

Adding these, we get $z+\frac{1}{z}=2\cos x$ and $z-\frac{1}{z}=2i\sin x$

By De Moivre's theorem, we have

$z^n+\frac{1}{z^n}=2\cos nx$ and $z^n-\frac{1}{z^n}=2i\sin^n x$

$\therefore$

$$\sin^m x\cos^n x=\frac{1}{(2i)^m}\cdot\frac{1}{2^n}\left(z+\frac{1}{z}\right)^n\left(z-\frac{1}{z}\right)^m$$

$$=\frac{1}{2^{m+n}}\cdot\frac{1}{i^m}\left(z+\frac{1}{z}\right)^n\left(z-\frac{1}{z}\right)^m$$

Using the Binomial theorem, extend each of the components on the R.H.S. Then, equidistant from the start and end, group the words. As a result, all such pairings may be expressed as the sines or cosines of various angles. Term by term, continue to integrate.

(v) If the sum of powers of $\sin x$ and $\cos x$ is an even negative integer, put $\tan x=z$.

Solved Examples:

1. Evaluate: $x^3+5x^2-4+\frac{7}{}+\frac{2}{\sqrt{}}\ dx$

Ans: $\int\left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right)dx$

$$=\int x^3dx+\int 5x^2dx-\int 4dx+\int\frac{7}{x}dx+\int\frac{2}{\sqrt{x}}dx$$

$$=\int x^3dx+5\cdot\int x^2dx-4\cdot\int 1\cdot dx+7\cdot\int\frac{1}{x}dx+2\cdot\int x^{-1/2}dx$$

$$=\frac{x^4}{4}+5\cdot\frac{x^3}{3}-4x+7\log|x|+2\left(\frac{x^{1/2}}{1/2}\right)+C$$

$$=\frac{x^4}{4}+\frac{5}{3}x^3-4x+7\log|x|+4\sqrt{x}+C$$

2. Evaluate: $\int \mathrm{e}^{x\log a}+\mathrm{e}^{a\log x}+\mathrm{e}^{a\log a}\mathrm{dx}$

Ans: We have,

$$\int e^{x\log a}+e^{a\log x}+e^{a\log a}dx$$

$$=\int e^{\log a^{x}}+e^{\log x^{a}}+e^{\log a^{a}}dx$$

$$=\int\left(a^{x}+x^{a}+a^{a}\right)dx$$

$$=\int a^{x}dx+\int x^{a}dx+\int a^{a}dx$$

$$=\frac{a^{x}}{\log a}+\frac{x^{a+1}}{a+1}+a^{a}\cdot x+C$$

3. Evaluate: $\int\frac{x^{4}}{x^{2}+1}dx$

Ans: $\int\frac{x^{4}}{x^{2}+1}dx$

$$=\int\frac{x^{4}-1+1}{x^{2}+1}dx=\int\frac{x^{4}-1}{x^{2}+1}+\frac{1}{x^{2}+1}dx$$

$$=\int\left(x^{2}-1\right)dx+\int\frac{1}{x^{2}+1}dx=\frac{x^{3}}{3}-x+\tan^{-1}x+C$$

4. Evaluate: $\int\frac{2^{x}+3^{x}}{5^{x}}dx$

Ans: $\int\frac{2^{x}+3^{x}}{5^{x}}dx$

$$=\int\left(\frac{2^{x}}{5^{x}}+\frac{3^{x}}{5^{x}}\right)dx$$

$$=\int\left[\left(\frac{2}{5}\right)^{x}+\left(\frac{3}{5}\right)^{x}\right]dx=\frac{(2/5)^{x}}{\log_{c}2/5}+\frac{(3/5)^{x}}{\log_{c}3/5}+C$$

5. Evaluate: $\int x^{3}\sin^{4}dx$

Ans: We have

$$I = \int x^3 \sin^4 dx$$

Let $x^4 = t \quad \Rightarrow \quad d\left(x^4\right) = dt$

$$\Rightarrow \quad 4x^3 dx = dt \Rightarrow \quad dx = \frac{1}{4x^3} dt$$