Revision Notes

Class – 12 Mathematics

Chapter 5 - Continuity and Differentiability

CONTINUITY

1. DEFINITION

A function $f(\text{x})$ is said to be continuous at $\text{x}=\text{a}$; where $\text{a}\in$ domain of $f(\text{x})$, if

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$$

i.e., $\text{LHL}=\text{RHL}=$ value of a function at $\text{x}=\text{a}$

or $\lim_{x\to a} f(x)=f(a)$

1.1 Reasons of discontinuity

If $f(\text{x})$ is not continuous at $\text{x}=\text{a}$, we say that $f(\text{x})$ is discontinuous at $x=a$

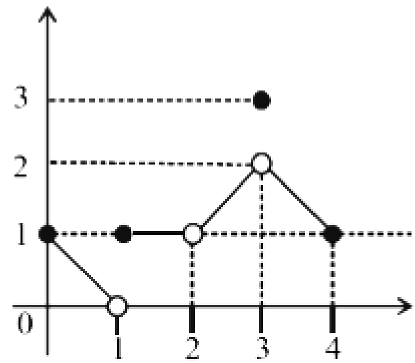
There are following possibilities of discontinuity:

1. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but they are not equal.

2. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists and are equal but not equal to $f(a)$

3. $f(a)$ is not defined.

4. At least one of the limits does not exist. The graph of the function will show a break at the location of discontinuity from a geometric standpoint.



The graph as shown is discontinuous at $x = 1, 2$ and 3 .

2. PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be continuous functions at $x = a$. Then,

1. $cf(x)$ is continuous at $x = a$, where c is any constant.

2. $f(x) \pm g(x)$ is continuous at $x = a$.

3. $f(x) \cdot g(x)$ is continuous at $x = a$.

4. $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.-

5. Assuming $f(x)$ be continuous on $[a, b]$ in such a way that the function $f(a)$ and $f(b)$ will be at opposite signs, then there will exists at least one solution of equation $f(x) = 0$ in the open interval (a, b)

3. THE INTERMEDIATE VALUE THEOREM

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, their exits a number c between a and b such that $f(c) = y_0$

The Function f, being continuous on (a, b) takes on every value between $f(a)$ and $f(b)$

Note:

That a function f which is continuous in [a, b] possesses the following properties:

(i) If $f(a)$ and $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x)=0$ in the open interval (a,b)

(ii) If K is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x)=K$ in the open interval (a,b)

4. CONTINUITY IN AN INTERVAL

(a) A function f is said to be continuous in (a, b) if f is continuous at each and every point $\in (a,b)$

(b) A function f is said to be continuous in a closed interval $[a,b]$ if :

(1) f is continuous in the open interval (a,b) and

(2) f is right continuous at 'a' i.e. $\text{Limit}_{x\to a^+} f(x)=f(a)=$ a finite quantity

(3) f is left continuous at 'b'; i.e. $\text{Limit}_{x\to b^-} f(x)=f(b)=a$ finite quantity

5. A LIST OF CONTINUOUS FUNCTIONS

Function f(x)	Interval in which f(x) is continuous
1. constant c	$(-\infty,\infty)$
2. x^n, n is an integer $\geqslant 0$	$(-\infty,\infty)$
3. x^{-n}, n is a positive integer	$(-\infty,\infty)-\{0\}$
4. $\lvert x-a\rvert$	$(-\infty,\infty)$
5. $P(x)=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n$	$(-\infty,\infty)$
6. $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty,\infty)-\{x;q(x)=0\}$
7. $\text{Sin}\,x$	$(-\infty,\infty)$
8. $\cos x$	$(-\infty,\infty)$

9. $\tan x$	$(-\infty,\infty)-\left\{(2\quad+1)\frac{}{2}:\quad\in\quad\right\}$
10. $\cot x$	$(-\infty,\infty)-\{n\pi : n\in I\}$
11. $\sec x$	$(-\infty,\infty)-\{(2n+1)$
12. $\operatorname{cos}ec\ x$	$\pi/2 : n\in I\}$
13. e^{x}	$(-\infty,\infty)-\{n\pi : n\in I\}$
14. $\log_c x$	$(-\infty,\infty)\ (0,\infty)$

6. TYPES OF DISCONTINUITIES

Type-1: (Removable type of discontinuities)

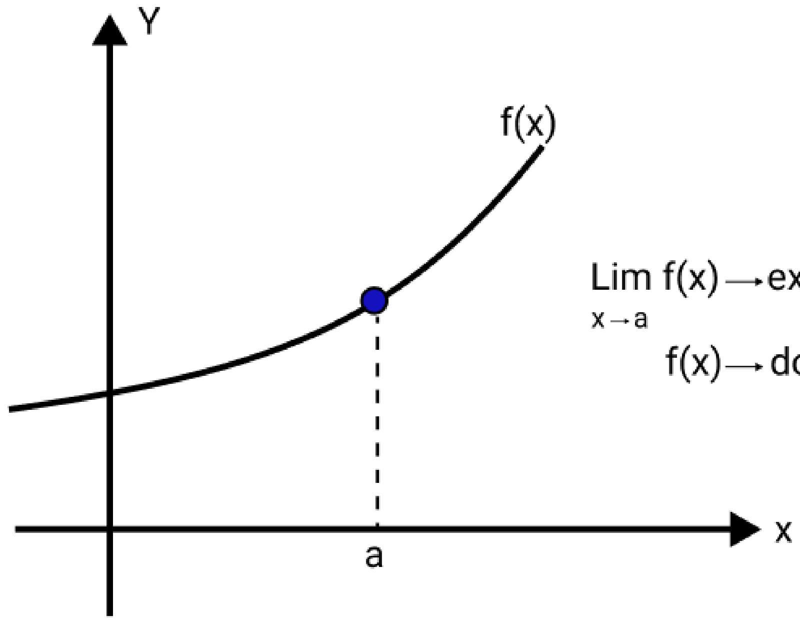
In this case, $\underset{x\to c}{\text{Limit}}\, f(\text{x})$ exists but it will not equal to $f(\text{c})$. As a result, the function is said to have a removable discontinuity or discontinuity of the first kind. In such scenario, we can redefine the function such that $\underset{x\to c}{\text{Limit}}\, f(x)=f(c)$ and make it continuous at $\text{x}=\text{c}$. It can be further categorised as:

(a) Missing Point Discontinuity:

Where $\underset{x\to a}{Limit}\, f(x)$ exists finitely but $f(a)$ is not defined.

E.g. $f(x)=\dfrac{(1-x)\left(9-x^2\right)}{(1-x)}$ will have a missing point discontinuity at $x=1$, and

$f(\text{x})=\dfrac{\sin \text{x}}{\text{x}}$ will have a missing point discontinuity at $\text{x}=0$



missing point discontinuity at x=a

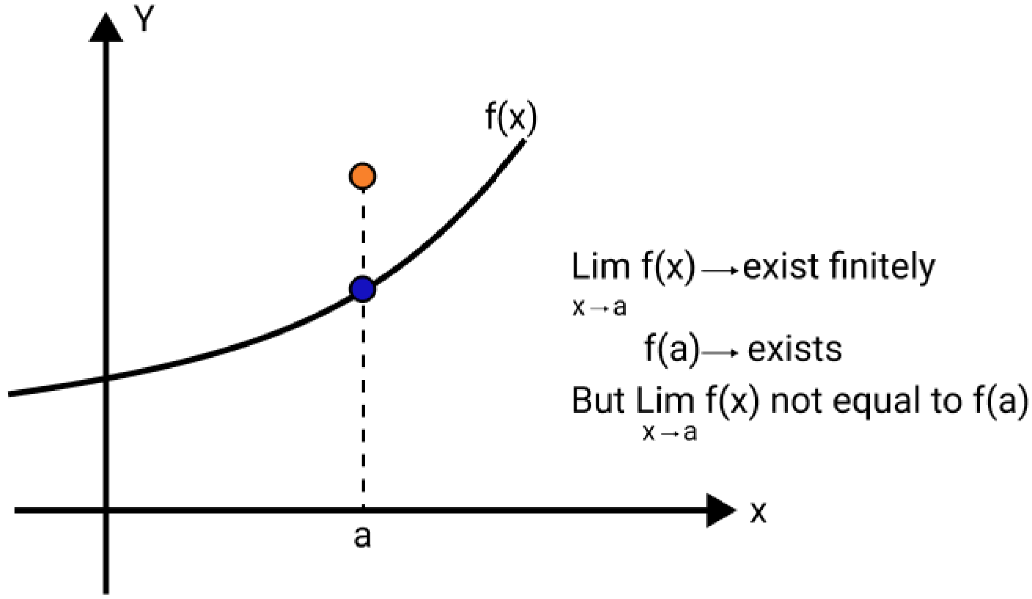
(b) Isolated Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}} f(x)$ exists f (a) also exists but;

$\text{Limit}_{x \to a} \neq f(\text{a})$

E.g. $f(x)=\dfrac{x^2-16}{x-4}, x \neq 4$ and $f(4)=9$ will have an isolated point discontinuity at $x=4$

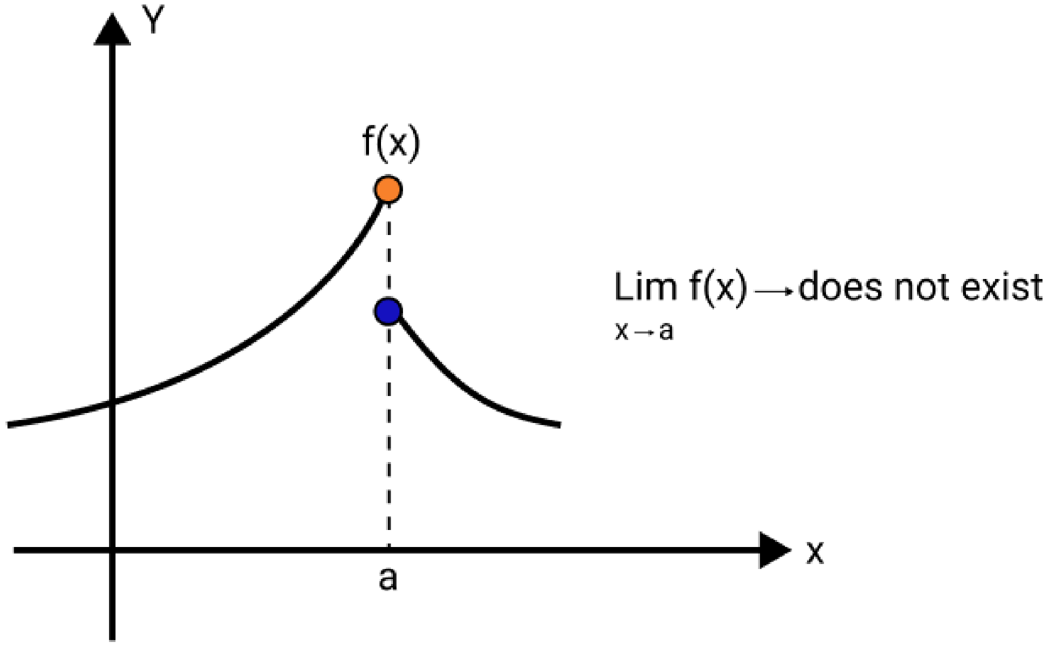
In the same way $f(x)=[x]+[-x]=\begin{bmatrix} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{bmatrix}$ will have an isolated point discontinuity at all $x \in I$.



Isolated point discontinuity at x=a

Type-2 : (Non-Removable type of discontinuities)

In case, $\underset{x\to a}{\text{Limit}}\, f(x)$ does not exist, then it is not possible to make the function continuous by redefining it. Such discontinuities are known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as:



non - removable discontinuity at x=a

(a) Finite Discontinuity:

E.g., $f(x)=x-[x]$ at all integral x; $f(x)=\tan^{-1}\frac{1}{x}$ at $x=0$ and $f(x)=\frac{1}{1+2^{\frac{1}{x}}}$ at $x=0$

(note that $f\left(0^{+}\right)=0;\ f\left(0^{-}\right)=1$) $1+2^{-}$

(b) Infinite Discontinuity:

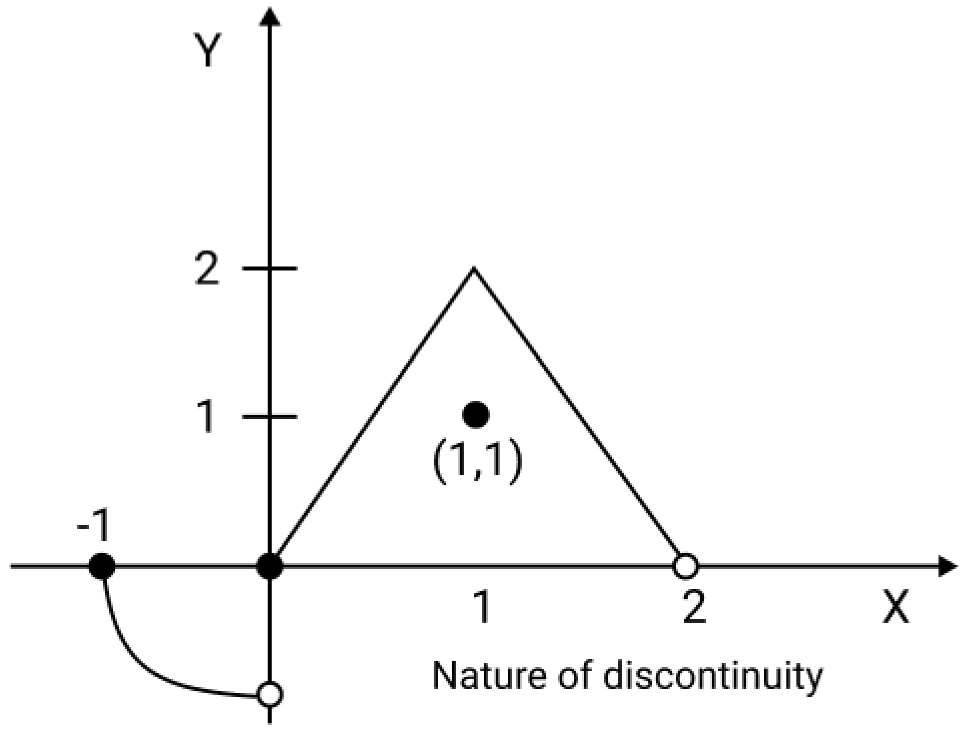
E.g., $f(x)=\frac{1}{x-4}$ or $g(x)=\frac{1}{(x-4)^{2}}$ at $x=4$; $f(x)=2^{\tan x}$

at $x=\frac{\pi}{2}$ and $f(x)=\frac{\cos x}{x}$ at $x=0$

(c) Oscillatory Discontinuity:

E.g., $f(x)=\sin\frac{1}{x}$ at $x=0$

In all these cases the value of $f(a)$ of the function at $x=a$ (point of discontinuity) may or may not exist but $\underset{x\to a}{\text{Limit}}$ does not exist.



From the adjacent graph note that

– f is continuous at $x=-1$

– f has isolated discontinuity at $x=1$

– f has missing point discontinuity at $x=2$

– f has non-removable (finite type) discontinuity at the origin.

Note:

(a) In case of dis-continuity of the second kind the nonnegative difference between the value of the RHL at $x=a$ and LHL at $x=a$ is called the jump of discontinuity. A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

(b) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.

(c) If $f(\mathrm{x})$ is continuous and $g(\mathrm{x})$ is discontinuous at $\mathrm{x}=\mathrm{a}$ then the product function $\phi(x)=f(x)\cdot g(x)$ is not necessarily be discontinuous at $x=a$. e.g.

$f(x)=x$ and $g(x)=\begin{cases}\sin\frac{\pi}{x} & x\neq 0\\ 0 & x=0\end{cases}$ (d) If $f(\mathrm{x})$ and $g(\mathrm{x})$ both are discontinuous at $\mathrm{x}=\mathrm{a}$ then the product function $\phi(x)=f(x)\cdot g(x)$ is not necessarily be discontinuous at $\mathrm{x}=\mathrm{a}$. e.g

$f(x)=-g(x)=\begin{cases}1 & x\geqslant 0\\ -1 & x<0\end{cases}$ (e) Point functions are to be treated as discontinuous

eg. $f(x)=\sqrt{1-x}+\sqrt{x-1}$ is not continuous at $x=1$

(f) A continuous function whose domain is closed must have a range also in closed interval.

(g) If f is continuous at $x=a$ and g is continuous at $\mathrm{x}=f$ (a) then the composite $g[f(\mathrm{x})]$ is continous at $\mathrm{x}=\mathrm{a}$

E.g $f(\mathrm{x})=\dfrac{\mathrm{x}\sin\mathrm{x}}{\mathrm{x}^2+2}$ and $g(\mathrm{x})=|\mathrm{x}|$ are continuous at $\mathrm{x}=0$, hence the composite $(gof)(\mathrm{x})=\left|\dfrac{\mathrm{x}\sin\mathrm{x}}{\mathrm{x}^2+2}\right|$ will also be continuous at $\mathrm{x}=0$.

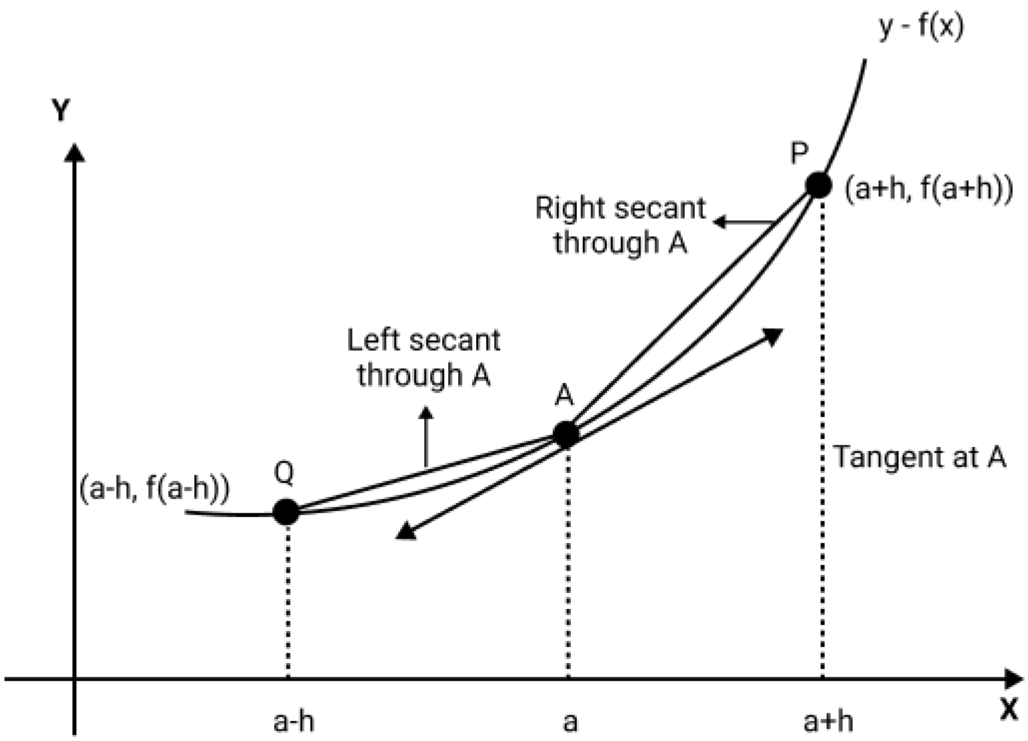
DIFFERENTIABILITY

1. DEFINITION

Let $f(\mathrm{x})$ bea real valued function defined on an open interval (a,b) where $c\in(a,b)$. Then $f(x)$ is said to be differentiable or derivable at $x=c$

if, $\lim\limits_{\mathrm{x}\to\mathrm{c}}\dfrac{f(\mathrm{x})-f(\mathrm{c})}{(\mathrm{x}-\mathrm{c})}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at $x=c$, and is denoted by $f'(\mathrm{c})$ or $\dfrac{\mathrm{d}}{\mathrm{dx}}(f(\mathrm{x}))_{\mathrm{x}=\mathrm{c}}$



- Slope of Right hand secant $=\dfrac{f(\mathrm{a}+\mathrm{h})-f(\mathrm{a})}{\mathrm{h}}$ as $\mathrm{h}\to 0, \mathrm{P}\to \mathrm{A}$ and secant $(\mathrm{AP})\to$ tangent at A

$\Rightarrow$ Right hand derivative $=\mathrm{Lim}_{\mathrm{h}\to 0}\left(\dfrac{f(\mathrm{a}+\mathrm{h})-f(\mathrm{a})}{\mathrm{h}}\right)$

$=$ Slope of tangent at A (when approached from right) $f'\left(\mathrm{a}^{+}\right)$

- Slope of Left hand secant $=\dfrac{f(\mathrm{a}-\mathrm{h})-f(\mathrm{a})}{-\mathrm{h}}$ as $\mathrm{h}\to 0, \mathrm{Q}\to \mathrm{A}$ and secant $\mathrm{AQ}\to$ tangent at A

$\Rightarrow$ Left hand derivative $=\mathrm{Lim}_{h\to 0}\left(\dfrac{f(a-h)-f(a)}{-h}\right)$

= Slope of tangent at A (when approached from left) $f'\left(\mathrm{a}^{-}\right)$

Thus, $f(\mathrm{x})$ is differentiable at $\mathrm{x}=\mathrm{c}$.

$\Leftrightarrow \lim\limits_{\to c} \dfrac{f() - f(c)}{(-c)}$ exists finitely

$\Leftrightarrow \lim\limits_{\to c^-} \dfrac{f() - f(c)}{(-c)} = \lim\limits_{\to c^+} \dfrac{f() - f(c)}{(-c)}$

$\Leftrightarrow \lim\limits_{h \to 0} \dfrac{f(c-h) - f(c)}{-h} = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

Hence, $\lim\limits_{x \to c^-} \dfrac{f(\mathbf{x}) - f(\mathbf{c})}{(\mathbf{x}-\mathbf{c})} = \lim\limits_{\mathbf{h} \to 0} \dfrac{f(\mathbf{c}-\mathbf{h}) - \mathbf{f}(\mathbf{c})}{-\mathbf{h}}$ is called the left hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^-)$ or $Lf'(c)$ While, $\lim\limits_{\mathbf{x} \to \mathbf{c}^+} \dfrac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{c})}{\mathbf{x}-\mathbf{c}} = \lim\limits_{\mathbf{h} \to 0} \dfrac{\mathbf{f}(\mathbf{c}+\mathbf{h}) - \mathbf{f}(\mathbf{c})}{\mathbf{h}}$ is called the right hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^+)$ or $Rf'(c)$

If $f'(c^-) \neq f'(c^+)$, we say that $f(x)$ is not differentiable at $x = c$.

2. DIFFERENTIABILITY IN A SET

1. A function $f(x)$ defined on an open interval (a,b) is said to be differentiable or derivable in open interval (a,b), if it is differentiable at each point of (a,b)

2. A function $f(x)$ defined on closed interval [a, b] is said to be differentiable or derivable. "If f is derivable in the open interval (a, b) and also the end points a and b, then f is said to be derivable in the closed interval [a, b]"

i.e., $\lim\limits_{\to a^+} \dfrac{f() - f(a)}{-a}$ and $\lim\limits_{\to b^-} \dfrac{f() - f(b)}{-b}$, both exist.

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

Note:

1. If $f(x)$ and $g(x)$ are derivable at $x =$ a then the functions $f(x)+g(x), f(x)-g(x), f(x)\cdot g(x)$ will also be derivable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$

2. If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x)\cdot g(x)$ can still be differentiable at $x = a$. E.g. $f(x) = x$ and $g(x) = |x|$

3. If $f(x)$ and g (x) both are not differentiable at $x=a$ then the product function; $F(x)=f(x)\cdot g(x)$ can still be differentiable at $x=$ a. E.g. $f(x)=|x|$ and $g(x)=|x|$

4. If $f(x)$ and $g(x)$ both are not differentiable at $x=a$ then the sum function $F(x)=f(x)+g(x)$ may be a differentiable function. E.g., $f(x)=|x|$ and $g(x)=-|x|$

5. If $f(x)$ is derivable at $x=a$

$\Rightarrow f'(x)$ is continuous at $x=a$.

e.g. $f(x)=\begin{bmatrix} 2 & \text{if } \neq 0 \\ 0 & \text{if } =0 \end{bmatrix}$

3. Relation B/W Continuity & Differentiability

We learned in the last section that if a function is differentiable at a point, it must also be continuous at that point, and therefore a discontinuous function cannot be differentiable. The following theorem establishes this fact.

Theorem: If a function is differentiable at a given point, it must be continuous at that same point. However, the inverse is not always true.

or $f(x)$ is differentiable at $x=c$

$\Rightarrow$ $f(x)$ is continuous at $x=c$

Converse: The reverse of the preceding theorem is not always true, i.e., a function might be continuous but not differentiable at a given point.

E.g., The function $f(x)=|x|$ is continuous at $x=0$ but it is not differentiable at $x=0$.

Note:

(a) Let $f'^{+}(a)=p; f'^{-}(a)=q$ where $p\ q$ are finite then

$\Rightarrow f$ is derivable at $x=a$

$\Rightarrow f$ is continuous at $x=a$

(ii) $p\neq q$ $\Rightarrow f$ is not derivable at $x=a$.

It is very important to note that f may be still continuous at $x = a$

In short, for a function f:

Differentiable $\Rightarrow$ Continuous;

Not Differentiable $\neq$ Not Continuous

(i.e., function may be continuous)

But,

Not Continuous $\Rightarrow$ Not Differentiable.

(b) If a function f is not differentiable but is continuous at $\mathbf{x} =$ a it geometrically implies a sharp corner at $\mathbf{x = a}$

Theorem 2: Let f and g be real functions such that fog is defined if g is continuous at $x = a$ and f is continuous at g

(a), show that fog is continuous at $x = a$.

DIFFERENTIATION:

1. DEFINITION

(a) Let us consider a function $y = f(x)$ defined in a certain interval. It has a definite value for each value of the independent variable x in this interval.

Now, the ratio of the function's increment to the independent variable's increment,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now, as $\Delta x \to 0, \Delta y \to 0$ and $\frac{\Delta y}{\Delta x} \to$ finite quantity, then derivative $f(x)$ exists and is denoted by y' or $f'(x)$ or $\frac{dy}{dx}$ Thus, $f'(x) = \lim_{x \to 0}\left(\frac{\Delta y}{x}\right) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{x}$ (if it exits) for the limit to exist,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

(Right Hand derivative) (Left Hand derivative)

(b) The derivative of a given function f at a point $x=a$ of its domain is defined as:

$\underset{h\to 0}{\text{Limit}}\frac{f(a+h)-f(a)}{h}$, provided the limit exists is denoted by $f'(a)$

Note that alternatively, we can define

$f'(a)=\text{Limit}_{x\to a}\frac{f(x)-f(a)}{x-a}$, provided the limit exists.

This method is called first principle of finding the derivative of $f(x)$

2. DERIVATIVE OF STANDARD FUNCTION

(i) $\frac{d}{dx}\left(x^n\right)=n\cdot x^{n-1}; x\in R, n\in R, x>0$

(ii) $\frac{d}{dx}\left(e^x\right)=e^x$

(iii) $\frac{d}{dx}\left(a^x\right)=a^x\cdot \ln a(a>0)$

(iv) $\frac{d}{dx}(\ln|x|)=\frac{1}{x}$

(v) $\frac{d}{dx}\left(\log_a|x|\right)=\frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x)=\cos x$

(vii) $\frac{d}{dx}(\cos x)=-\sin x$

(viii) $\frac{d}{dx}(\tan x)=\sec^2 x$

(ix) $\frac{d}{dx}(\sec x)=\sec x\cdot\tan x$

(x) $\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cdot\cot x$

(xi) $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(\text{ constant })=0$

(xiii) $\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}},\quad -1<x<1$

(xiv) $\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}},\quad -1<x<1$

(xv) $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2},\quad x\in R$

(xvi) $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2},\quad x\in R$

(xvii) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}},\quad |x|>1$

(xviii) $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}},\quad |x|>1$

(xix) Results:

If the inverse functions $f(g)$ are defined by $y=f(x); x=g(y)$. Then

$g(f(x))=x \;\Rightarrow\; g'(f(x))\cdot f'(x)=1$

This result can also be written as, if $\frac{dy}{dx}$ exists and $\frac{dy}{dx}\neq 0$, then $\frac{dx}{dy}=1/\left(\frac{dy}{dx}\right)$ or $\frac{dy}{dx}\cdot\frac{dx}{dy}=1$ or $\frac{dy}{dx}=1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy}\neq 0\right]$

3. THEOREMS ON DERIVATIVES

If u and v are derivable functions of x, then,

(i) Term by term differentiation : $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) Multiplication by a constant $\frac{d}{dx}(Ku) = K\frac{du}{dx}$, where K is any constant

(iii) "Product Rule" $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$ known as In general,

(a) If $u_1, u_2, u_3, u_4, \ldots, u_n$ are the functions of x, then

$$\frac{d}{dx}(u_1 \cdot u_2 \cdot u_3 \cdot u_4 \ldots . u_n)$$

$$= \left(\frac{du_1}{dx}\right)(u_2 u_3 u_4 \ldots u_n) + \left(\frac{du_2}{dx}\right)(u_1 u_3 u_4 \ldots u_n)$$

$$+\left(\frac{du_3}{dx}\right)(u_1 u_2 u_4 \ldots u_n) + \left(\frac{du_4}{dx}\right)(u_1 u_2 u_3 u_5 \ldots u_n)$$

$$+ \ldots + \left(\frac{du_n}{dx}\right)(u_1 u_2 u_3 \ldots u_{n-1})$$

(iv) Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$ where $v \neq 0$ known as

(b) Chain Rule : If $y = f(u), u = g(w), w = h(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$

or $\frac{dy}{dx} = f'(u) \cdot g'(\quad) \cdot h'(x)$

Note:

In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}$

4. METHODS OF DIFFERENTIATION

4.1 Derivative by using Trigonometrical Substitution

The use of trigonometrical transforms before differentiation greatly reduces the amount of labour required. The following are some of the most significant findings:

(i) $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}, \tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$

(iv) $\sin 3x = 3\sin x - 4\sin^3 x$

(v) $\cos 3x = 4\cos^3 x - 3\cos x$

(vi) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

(vii) $\tan\left(\dfrac{\pi}{4}+x\right) = \dfrac{1+\tan x}{1-\tan x}$

(viii) $\tan\left(\dfrac{\pi}{4}-x\right) = \dfrac{1-\tan x}{1+\tan x}$

(ix) $\sqrt{(1\pm\sin x)} = \left|\cos\dfrac{x}{2}\pm\sin\dfrac{x}{2}\right|$

(x) $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left(\dfrac{x\pm y}{1\mp xy}\right)$

(xi) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(xii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$

(xiii) $\sin^{-1}x + \cos^{-1}x = \tan^{-1}x + \cot^{-1}x = \sec^{-1}x + \text{cosec}^{-1}x = \pi/2$

(xiv) $\sin^{-1}x = \text{cosec}^{-1}(1/x); \cos^{-1}x = \sec^{-1}(1/x); \tan^{-1}x = \cot^{-1}(1/x)$

Note:

Some standard substitutions:

Expressions **Substitutions**

$\sqrt{\left(a^2-x^2\right)}$ $x=a\sin\theta$ or $a\cos\theta$

$\sqrt{\left(a^2+x^2\right)}$ $x=a\tan\theta$ or $a\cot\theta$

$\sqrt{\left(x^2-a^2\right)}$ $x=a\sec\theta$ or $a\operatorname{cosec}\theta$

$\sqrt{\left(\frac{a+x}{a-x}\right)}$ or $\sqrt{\left(\frac{a-x}{a+x}\right)}$ $x=a\cos\theta$ or $a\cos 2\theta$

$\sqrt{(a-x)(x-b)}$ or $x=a\cos^2\theta+b\sin^2\theta$

$\sqrt{\left(\frac{a-x}{x-b}\right)}$ or $\sqrt{\left(\frac{x-}{a-x}\right)}$

$\sqrt{(x-a)(x-b)}$ or $x=a\sec^2\theta-b\tan^2\theta$

$\sqrt{\left(\frac{x-a}{x-b}\right)}$ or $\sqrt{\left(\frac{x-}{x-a}\right)}$

$\sqrt{\left(2ax-x^2\right)}$ $x=a(1-\cos\theta)$

4.2 Logarithmic Differentiation

To find the derivative of:

If $y=\{f_1(x)\}^{f_2(x)}$ or $y=f_1(x)\cdot f_2(x)\cdot f_3(x)\ldots$

or $y=\frac{f_1(x)\cdot f_2(x)\cdot f_3(x)\ldots}{g_1(x)\cdot g_2(x)\cdot g_3(x)\ldots}$ then it's easier to take the function's logarithm first and then differentiate. This is referred to as the logarithmic function's derivative.

Important Notes (Alternate methods)

1. If $y=\{f(x)\}^{g(x)}=e^{g(x)\ln f(x)}\left((\text{ variable })^{\text{varable}}\right)\left\{\because x=e^{\ln x}\right\}$

$$\therefore \frac{dy}{dx} = e^{g(x)\ln f(x)} \cdot \left\{ g(x) \cdot \frac{d}{dx} \ln f(x) + \ln f(x) \cdot \frac{d}{dx} g(x) \right\}$$

$$= \{f(x)\}^{g(x)} \cdot \left\{ g(x) \cdot \frac{f'(x)}{f(x)} + \ln f(x) \cdot g'(x) \right\}$$

2. If $y = \{f(x)\}^{g(x)}$

$\therefore \frac{dy}{dx} =$ Derivative of y treating $f(x)$ as constant + Derivative of y treating $g(x)$ as constant

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot \frac{d}{dx} g(x) + g(x)\{f(x)\}^{g(x)-1} \cdot \frac{d}{dx} f(x)$$

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot g'(x) + g(x) \cdot \{f(x)\}^{g(x)-1} \cdot f'(x)$$

4.3 Implicit Differentiation: $\phi(x, y) = 0$

(i) To get dy/dx with the use of implicit function, we differentiate each term w.r.t. x, regarding y as a function of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit function, both x and y are present.

Alternate Method: If $f(x, y) = 0$

then $$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{\text{diff of } f \text{ w.r.t. } x \text{ treating } y \text{ as constant}}{\text{diff . of } f \text{ w.r.t. } y \text{ treating } x \text{ as constant}}$$

4.4 Parametric Differentiation

If $y = f(t); x = g(t)$ where t is a Parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Note:

1. $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

2. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}\left(\because \frac{dy}{dx}\right.$ in terms of t)

$=\frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)\cdot\frac{1}{f'(t)}$ { From (1)}

$=\frac{f''(t)g'(t)-g''(t)f'(t)}{\left\{f'(t)\right\}}$

4.5 Derivative of a Function w.r.t. another Function

Let $y=f(x); z=g(x)$ then $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{f'(x)}{g'(x)}$

4.6 Derivative of Infinite Series

When one or more terms are removed from an infinite series, the series stays unaltered. as a result.

(A) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+......\infty}}}$

then $y=\sqrt{f(x)+y}\Rightarrow\left(y^2-y\right)=f(x)$

Differentiating both sides w.r.t. x, we get $(2y-1)\frac{dy}{dx}=f'(x)$

(B) If $y=\{f(x)\}^{\{f(x)\}^{\{f(x)\}^{-1}}}$ then $y=\{f(x)\}^y\Rightarrow y=e^{y\ln f(x)}$

Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx}=\frac{y\{f(x)\}^{y-1}\cdot f'(x)}{1-\{f(x)\}^y\cdot\ell n\, f(x)}=\frac{y^2f'(x)}{f(x)\{1-y\ell nf(x)\}}$

5. Derivative of Order Two & Three

Let us assume a function $y=f(x)$ be defined on an open interval (a,b). It's derivative, if it exists on (a,b), is a certain function $f'(x)\left[\right.$ or (dy/dx) or $\left.y'\right]$ is called the first derivative of y w.r.t. x. If it occurs that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x is denoted by $f''(x)$ or $\left(d^2y/dx^2\right)$ or y''.

Similarly, the 3^{rd} order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ it is also denoted by $f''(x)$ or y''' Some Standard Results :

(i) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}\cdot a^n \cdot (ax+b)^{m-n}, m \geqslant n$

(ii) $\frac{d^n}{dx^n} x^n = n!$

(iii) $\frac{d^n}{dx^n}\left(e^{mx}\right) = m^n \cdot e^{mx}, m \in R$

(iv) $\frac{d^n}{dx^n}(\sin(ax+b)) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right), n \in N$

(v) $\frac{d^n}{dx^n}(\cos(ax+b)) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right), n \in N$

(vi) $\frac{d^n}{dx^n}\left\{e^{ax}\sin(bx+c)\right\} = r^n \cdot e^{ax} \cdot \sin(bx+c+n\phi), n \in N$

where $r = \sqrt{\left(a^2+b^2\right)}, \phi = \tan^{-1}(b/a)$

(vii) $\frac{d^n}{dx^n}\left\{e^{ax} \cdot \cos(bx+c)\right\} = r^n \cdot e^{ax} \cdot \cos(bx+c+n\phi), n \in N$

where $r = \sqrt{\left(a^2+b^2\right)}, \phi = \tan^{-1}(b/a)$

6. DIFFERENTIATION OF DETERMINANTS

If $F(X) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$ where $f, g, h, \ell, m, n, u, v, w$ are differentiable function of x

then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

7. L' HOSPITAL'S RULE

If $f(x)$ and $g(x)$ are functions of x such that :

(i) $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)=\infty=\lim_{x\to a} g(x)$ $f(x)$ and

(ii) Both $f(x)$ and $g(x)$ are continuous at $x=a$ and

(iii) Both $f(x)$ and $g(x)$ are differentiable at $x=a$ and

(iv) Both $f(x)$ and $g(x)$ are continuous at $x=a$, Then $\text{Limit}_{x\to a}\frac{f(x)}{g(x)}=\text{Limit}_{x\to a}\frac{f'(x)}{g'(x)}=\text{Limit}_{x\to a}\frac{f''(x)}{g''(x)}$ & so on till determinant form vanishes.