Revision Notes

Class - 12 Maths

Chapter 2 - Inverse Trigonometric Functions

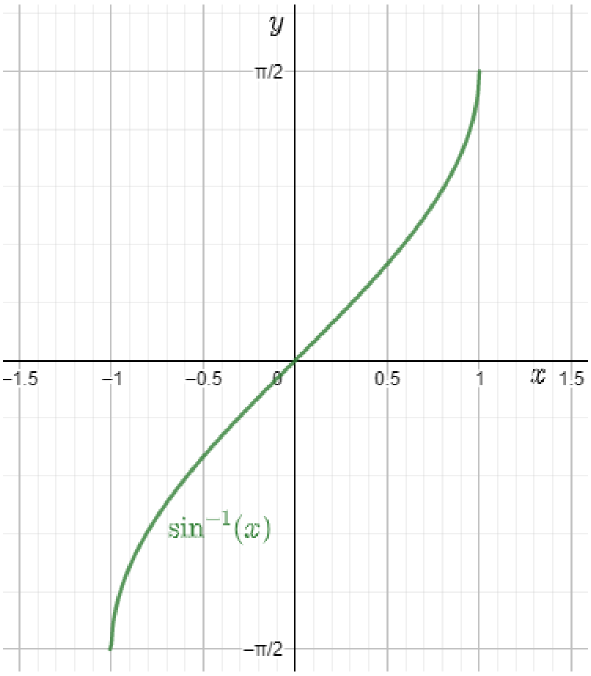
Domain and range of all inverse trigonometric functions

Function	Domain	Range
1. $y=\sin^{-1}x$ if $x=\sin y$	$-1\le x\le 1$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
2. $y=\cos^{-1}x$ if $x=\cos y$	$-1\le x\le 1$	$[0,\pi]$
3. $y=\tan^{-1}x$ if $x=\tan y$	$-\infty<x<\infty$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
4. $y=\cot^{-1}x$ if $x=\cot y$	$-\infty<x<\infty$	$(0,\pi)$
5. $y=\text{cosec}^{-1}x$ if $x=\text{cosec}y$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
6. $y=\sec^{-1}x$ if $x=\sec y$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$

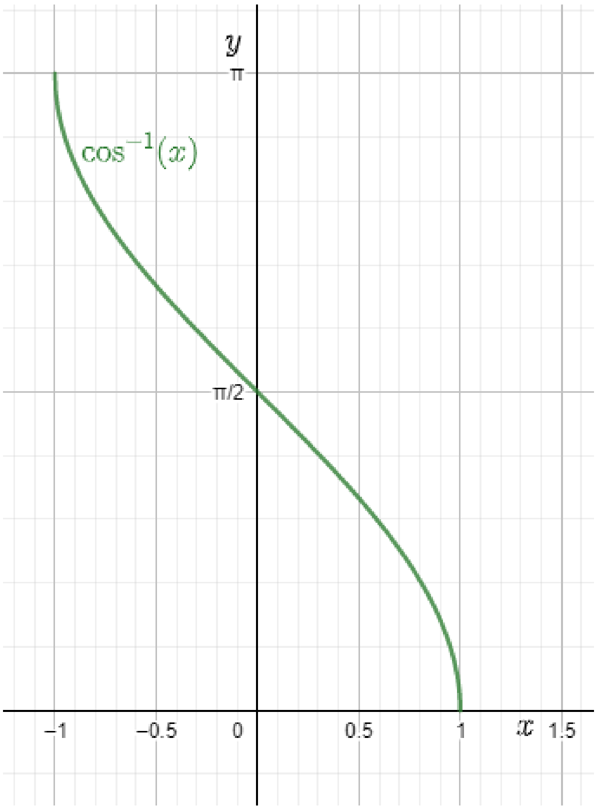
- We must note that inverse trigonometric functions cannot be expressed in terms of trigonometric functions as their reciprocals. For example, $\sin^{-1}x\ne\frac{1}{\sin x}$.
- The **principal value** of a trigonometric function is that value which lies in the range of principal branch.
- The functions $\sin^{-1}x$ & $\tan^{-1}x$ are increasing functions in their domain.
- The functions $\cos^{-1}x$ & $\cot^{-1}x$ are decreasing functions in over domain.

Graphs of inverse trigonometric functions

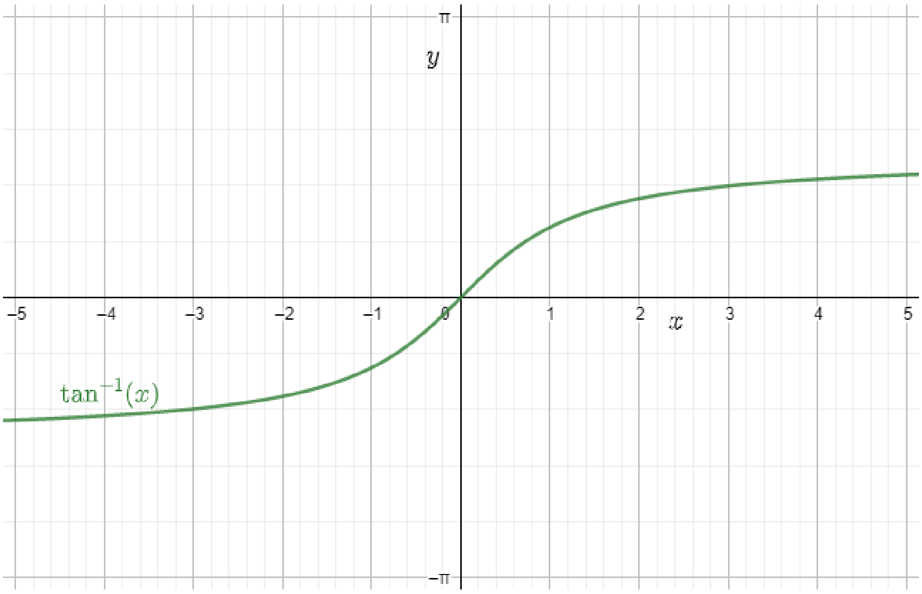
a) Graph of $\sin^{-1}x$ is shown below,



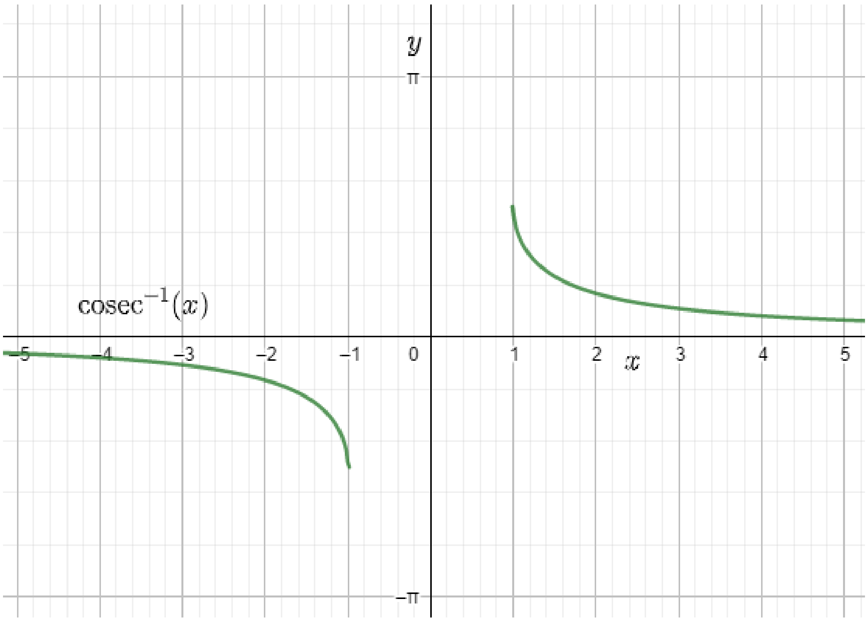
b) Graph of $\cos^{-1} x$ is shown below,



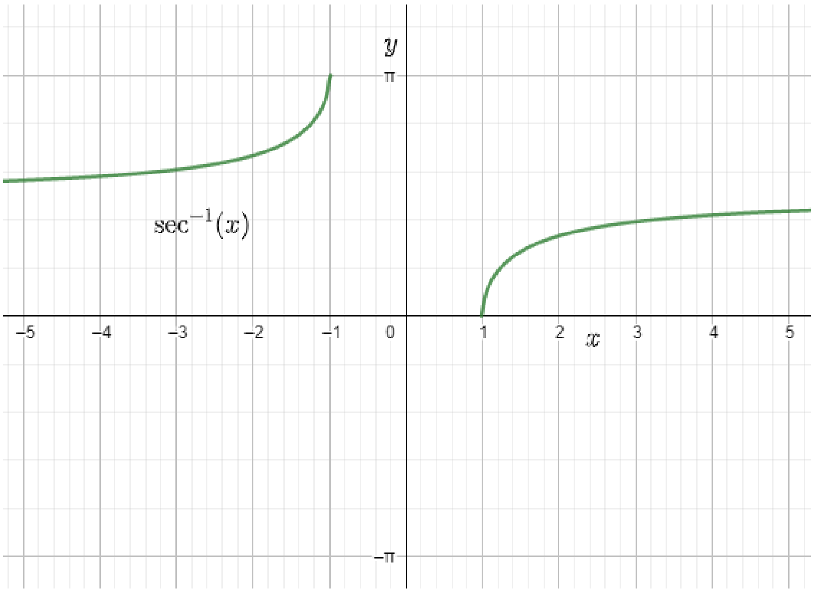
c) Graph of $\tan^{-1} x$ is shown below,



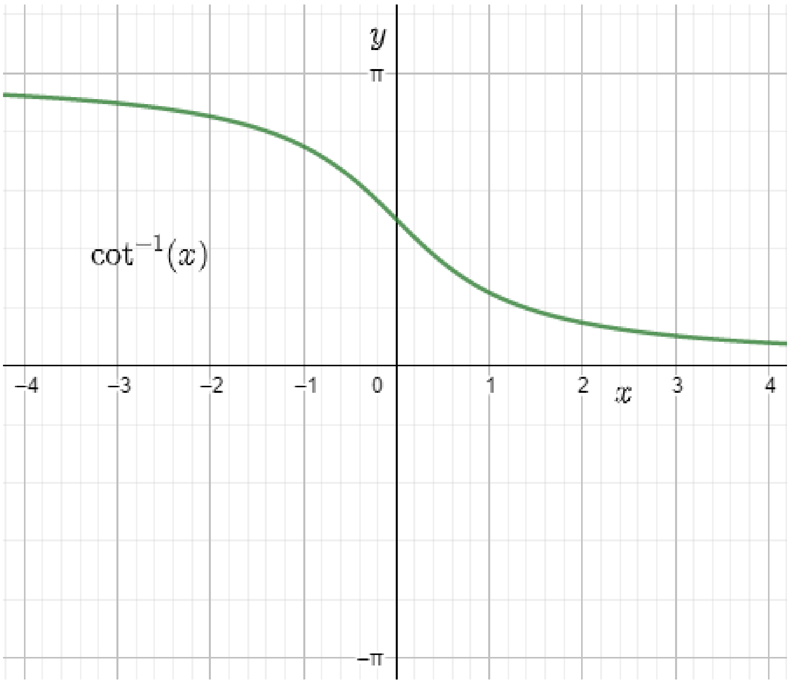
d) Graph of $\operatorname{cosec}^{-1} x$ is shown below,



e) Graph of $\sec^{-1} x$ is shown below,



f) Graph of $\cot^{-1} x$ is shown below,



Properties of inverse trigonometric functions

1. Property I

a) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$**, for all** $x \in (-\infty, 1] \cup [1, \infty)$

Let us prove this by considering $\operatorname{cosec}^{-1}x = \theta$ ……(i)

Taking cosec on both sides,

$x = \operatorname{cosec}\theta$

Using reciprocal identity,

$$\Rightarrow \frac{1}{x} = \sin\theta$$

$$\{\because x \in (-\infty, -1] \cup [1, \infty)\} \Rightarrow \frac{1}{x} \in [-1, 1]\{0\}$$

$$\operatorname{cosec}^{-1}x = \theta \Rightarrow \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{x}\right) \quad \text{……(ii)}$$

From (i) and (ii), we get

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$$

Hence proved.

b) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$**, for all** $x \in (-\infty, 1] \cup [1, \infty)$

Let us prove this by taking $\sec^{-1}x = \theta$ ……(i)

Taking sec on both sides,

$\Rightarrow x = \sec\theta$

Using reciprocal identity,

$$\Rightarrow \frac{1}{x} = \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{x}\right) \quad \text{……(ii)}$$

Then, $x \in (-\infty, 1] \cup [1, \infty)$ and $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$$\begin{cases} \because x = (-\infty, -1] \cup [1, \infty) \\ \Rightarrow \frac{1}{x} \in [-1, 1] - \{0\} \text{ and } \theta \in [0, \pi] \end{cases}$$

From (i) and (ii), we get

$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x)$$

Hence proved.

c) $\mathbf{tan}^{-1}\left(\frac{1}{\mathbf{x}}\right) = \begin{cases} \mathbf{cot}^{-1}\mathbf{x}, \ \mathbf{for}\ \mathbf{x} > 0 \\ -\pi + \mathbf{cot}^{-1}\mathbf{x}, \ \mathbf{for}\ \mathbf{x} < 0 \end{cases}$

Let us prove this by taking $\cot^{-1} x = \theta$. Then $x \in R, x \neq 0$ and $\theta \in [0, \pi]$ ……(i)

Now there are two cases that arise:

Case I: When $x > 0$

In this case, we have $\theta \in \left(0, \frac{\pi}{2}\right)$

Considering $\cot^{-1} x = \theta$

Taking $\cot$ on both sides,

$\Rightarrow x = \cot\theta$

Using reciprocal property,

$$\Rightarrow \frac{1}{x} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{1}{x}\right) \quad \text{……(ii)}$$

From (i) and (ii), we get $\left\{\because \theta \in \left(0, \frac{\pi}{2}\right)\right\}$

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x, \text{ for all } x > 0$$

Case II: When $x < 0$

$$\theta \in \left(- \;\; \pi \right) \qquad \because \;\; = \qquad \theta <$$

Now, $\frac{\pi}{2} < \theta < \pi$

$\Rightarrow -\frac{\pi}{2} < \theta - \pi < 0$

$\Rightarrow \theta - \pi \in \left(-\frac{\pi}{2}, 0\right)$

$\therefore \cot^{-1} x = \theta$

Taking cot on both sides,

$\Rightarrow x = \cot\theta$

Using reciprocal property,

$\Rightarrow \frac{1}{x} = \tan\theta$

$\Rightarrow \frac{1}{x} = -\tan(\pi - \theta)$

$\Rightarrow \frac{1}{x} = \tan(\theta - \pi) \quad \{\because \tan(\pi - \theta) = -\tan\theta\}$

$\Rightarrow \theta - \pi = \tan^{-1}\left(\frac{1}{x}\right) \quad \left\{\because \theta - \pi \in \left(-\frac{\pi}{2}, 0\right)\right\}$

$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = -\pi + \theta \quad \ldots\ldots(iii)$

From (i) and (iii), we get

$\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1} x$, if $x < 0$

Hence it is proved that $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

2. Property II

a) $\sin^{-1}(-x) = -\sin^{-1}(x)$**, for all** $x \in [-1, 1]$

b) $\tan^{-1}(-x) = -\tan^{-1} x$**, for all** $x \in R$

c) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$**, for all** $x \in (-\infty, -1] \cup [1, \infty)$

Clearly, $-x \in [-1,1]$ for all $x \in [-1,1]$

Let us prove a) by taking $\sin^{-1}(-x) = \theta$

Then, taking $\sin$ on both sides, we get

$-x = \sin\theta \quad \ldots\ldots(i)$

$\Rightarrow x = -\sin\theta$

$\Rightarrow x = \sin(-\theta)$

$\Rightarrow -\theta = \sin^{-1} x$

$\left\{\because x \in [-1,1] \text{ and } -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ for all } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$

$\Rightarrow \theta = -\sin^{-1} x \quad \ldots\ldots(ii)$

From (i) and (ii), we get

$\sin^{-1}(-x) = -\sin^{-1}(x)$

Hence proved.

The b) and c) properties can also be proved in the similar manner.

3. Property III

a) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$**, for all** $x \in [-1,1]$

b) $\sec^{-1}(-x) = \pi - \sec^{-1} x$**, for all** $x \in (-\infty, -1] \cup [1, \infty)$

c) $\cot^{-1}(-x) = \pi - \cot^{-1} x$**, for all** $x \in R$

Clearly, $-x \in [-1,1]$ for all $x \in [-1,1]$

Let us prove it by taking $\cos^{-1}(-x) = \theta \quad \ldots\ldots(i)$

Then, taking $\cos$ on both sides, we get

$-x = \cos\theta$

$\Rightarrow x = -\cos\theta$

$\Rightarrow x = \cos(\pi - \theta)$

$\{\because x \in [-1,1] \text{ and } \pi - \theta \in [0, \pi] \text{ for all } \theta \in [0, \pi]\}$

$\cos^{-1} x = \pi - \theta$

$\Rightarrow \theta = \pi - \cos^{-1} x \quad \ldots\ldots(ii)$

From (i) and (ii), we get

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

Hence Proved.

The b) and c) properties can also be proved in the similar manner.

4. Property IV

a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$**, for all** $x \in [-1,1]$

Let us prove it by taking $\sin^{-1} x = \theta$ ......(i)

Then, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad [\because x \in [-1,1]]$

$\Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \le -\theta \le \frac{\pi}{2}$

$\Rightarrow 0 \le \frac{\pi}{2} - \theta \le \pi$

$\Rightarrow \frac{\pi}{2} - \theta \in [0, \pi]$

Now we consider $\sin^{-1} x = \theta$

Taking $\sin$ on both sides, we get

$\Rightarrow x = \sin\theta$

Changing functions, we get

$\Rightarrow x = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$

$\left\{\because x \in [-1,1] \text{ and } \left(\frac{\pi}{2} - \theta\right) \in [0, \pi]\right\}$

$\Rightarrow \theta + \cos^{-1} x = \frac{\pi}{2}$ ......(ii)

From (i) and (ii), we get

$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

Hence proved.

b) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$**, for all** $x\in R$

Let us prove it by taking $\tan^{-1}x=\theta$ ……(i)

Then, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ $\{\because x\in R\}$

$\Rightarrow -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2}<-\theta<\frac{\pi}{2}$

$\Rightarrow 0<\frac{\pi}{2}-\theta<\pi$

$\Rightarrow\left(\frac{\pi}{2}-\theta\right)\in(0,\pi)$

Now consider $\tan^{-1}x=\theta$

Taking $\tan$ on both sides, we get

$\Rightarrow x=\tan\theta$

$\Rightarrow x=\cot\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \cot^{-1}x=\frac{\pi}{2}-\theta$ $\left\{\because \frac{\pi}{2}-\theta\in(0,\pi)\right\}$

$\Rightarrow \theta+\cot^{-1}x=\frac{\pi}{2}$ ……(ii)

From (i) and (ii), we get

$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

c) $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$**, for all** $x\in(-\infty,-1]\cup[1,\infty)$

Let us prove it by taking $\sec^{-1}x=\theta$ ……(i)

Then, $\theta \in [0,\pi]-\left\{\frac{\pi}{2}\right\}$ $\{\because x \in (-\infty,-1] \cup [1,\infty)\}$

$\Rightarrow 0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$

$\Rightarrow -\pi \le -\theta \le 0, \theta \ne \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \le \frac{\pi}{2}-\theta \le \frac{\pi}{2}, \frac{\pi}{2}-\theta \ne 0$

$\Rightarrow \left(\frac{\pi}{2}-\theta\right) \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \frac{\pi}{2}-\theta \ne 0$

Now considering $\sec^{-1} x = \theta$

Taking sec on both sides, we get

$\Rightarrow x = \sec\theta$

$\Rightarrow x = \operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \operatorname{cosec}^{-1} x = \frac{\pi}{2}-\theta$

$\left\{\because \left(\frac{\pi}{2}-\theta\right) \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \frac{\pi}{2}-\theta \ne 0\right\}$

$\Rightarrow \theta + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$ ....(ii)

From (i) and (ii), we get

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

5. Property V

a) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, xy < 1$

b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}, xy > -1$

c) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy > 1; x, y = 0$

Let us prove a) by taking $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$.

Taking tan on both sides for both terms, we get $x = \tan\theta$ and $y = \tan\phi$.

Using formula for $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$, we can write

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

Writing in terms of x and y,

$$\tan(\theta+\phi)=\frac{x+y}{1-xy}$$

$$\Rightarrow \theta+\phi=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Therefore $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}, xy<1$.

Hence proved.

The properties b) and c) can be proved in similar manner by considering y as $-y$ and y as x respectively in the above proof.

6. Property VI

a) $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}, |x|\le 1$

b) $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}, x\ge 0$

c) $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}, -1<x<1$

Let us prove a) by taking $\tan^{-1}x=y$.

Taking $\tan$ on both sides, we get

$x=\tan y$

We can write $\sin^{-1}\frac{2x}{1+x^2}$ as $\sin^{-1}\frac{2\tan y}{1+\tan^2 y}$.

Using formula $\sin 2x=\frac{2\tan x}{1+\tan^2 x}$, we get

$$\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}(\sin 2y)$$

Using $\sin^{-1}(\sin x)=x$, this can be written as

$$\sin^{-1}\frac{2x}{1+x^2}=2y$$

$$\Rightarrow \sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x$$

Hence proved.

The same process can be followed to prove properties b) and c) as well.

7. Property VII

a) $\sin\left(\sin^{-1}x\right) = x$**, for all** $x \in [-1,1]$

b) $\cos\left(\cos^{-1}x\right) = x$**, for all** $x \in [-1,1]$

c) $\tan\left(\tan^{-1}x\right) = x$**, for all** $x \in R$

d) $\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right) = x$**, for all** $x \in (-\infty,-1] \cup [1,\infty)$

e) $\sec\left(\sec^{-1}x\right) = x$**, for all** $x \in (-\infty,-1] \cup [1,\infty)$

f) $\cot\left(\cot^{-1}x\right) = x$**, for all** $x \in R$

Let us prove a). We know that, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ exists such that $fof^{-1}(y) = f\left(f^{-1}(y)\right) = y$ for all $y \in B$.

Clearly, all these results are direct consequences of this property.

Aliter: Let $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \in [-1,1]$ such that $\sin\theta = x$.

Taking $\sin$ on both sides, $\theta = \sin^{-1}x$

$\therefore x = \sin\theta = \sin\left(\sin^{-1}x\right)$

Hence, $\sin\left(\sin^{-1}x\right) = x$ for all $x \in [-1,1]$ and we proved it.

We can prove properties from b) to f) in a similar manner.

It should be noted that, $\sin^{-1}(\sin\theta) \neq \theta$, if $\notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let us understand this better. The function $y = \sin^{-1}(\sin x)$ is periodic and has period 2π.

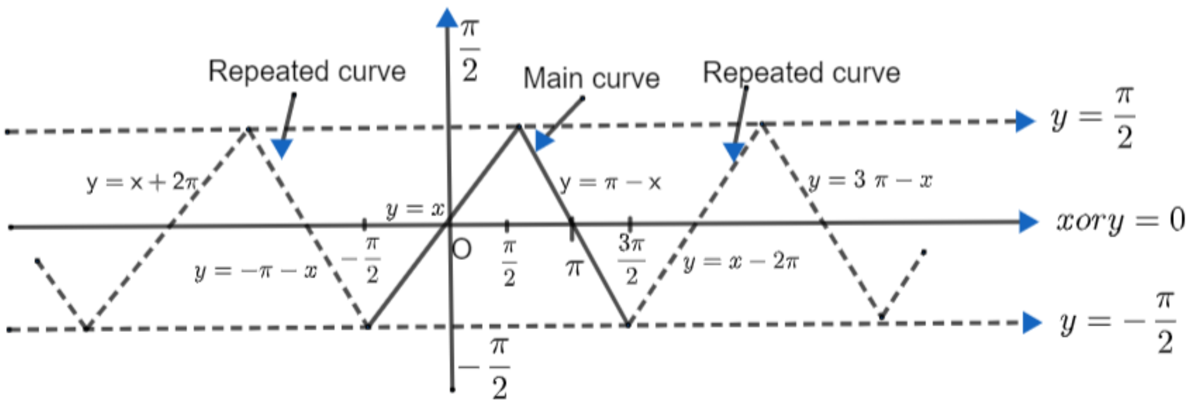
To draw this graph, we should draw the graph for one interval of length 2π and repeat the entire values of x.

As we know,

$$\sin^{-1}(\sin x)=\begin{cases} x; & -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ (\pi-x); & -\frac{\pi}{2}\le \pi-x<\frac{\pi}{2}\left(\text{i.e.}, \frac{\pi}{2}\le x\le\frac{3\pi}{2}\right)\end{cases}$$

$$\Rightarrow \sin^{-1}(\sin x)=\begin{cases} x, & -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ \pi-x, & \frac{\pi}{2}\le x\le\frac{3\pi}{2},\end{cases}$$

This is plotted as



Thus, we can note that the graph for $y=\sin^{-1}(\sin x)$ is a straight line up and a straight line down with slopes 1 and -1 respectively lying between $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

The below result for the definition of $\sin^{-1}(\sin x)$ must be kept in mind.

$$y=\sin^{-1}(\sin x)=\begin{cases} x+2\pi; & -\frac{5\pi}{2}\le x\le-\frac{3\pi}{2} \\ -\pi-x; & -\frac{3\pi}{2}\le x\le-\frac{\pi}{2} \\ x; & -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ \pi-x; & \frac{\pi}{2}\le x\le\frac{3\pi}{2} \\ x-2\pi; & \frac{3\pi}{2}\le x\le\frac{5\pi}{2} \text{ ...and so on}\end{cases}$$

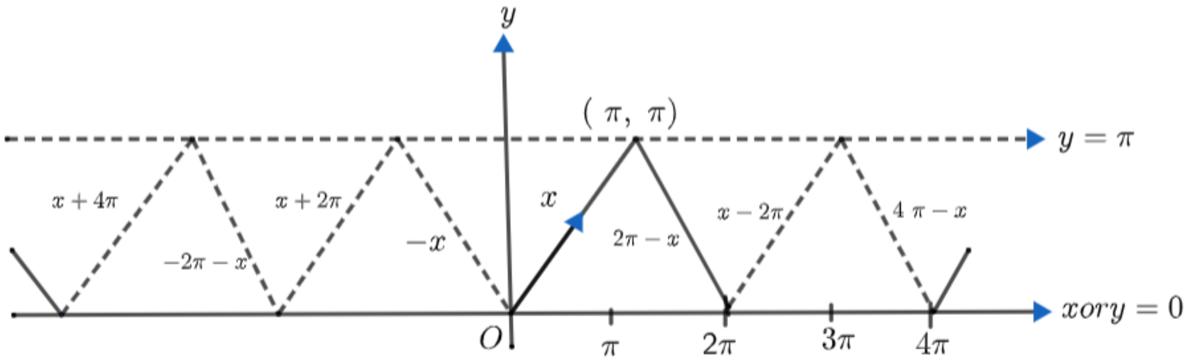
Now we consider $y = \cos^{-1}(\cos x)$ which is periodic and has period 2π.

To draw this graph, we should draw the graph for one interval of length 2π and repeat the entire values of x of length 2π
As we know,

$$\cos^{-1}(\cos x) = \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & 0 \le 2\pi - x \le \pi, \end{cases}$$

$$\Rightarrow \cos^{-1}(\cos x) = \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & \pi \le x \le 2\pi, \end{cases}$$

Thus, it has been defined for $0 < x < 2\pi$ that has length 2π.
So, its graph could be plotted as;



Thus, the curve $y = \cos^{-1}(\cos x)$ and we can not the results as

$$\cos^{-1}(\cos x) = \begin{cases} -x, & \text{if } x \in [-\pi, 0] \\ x, & \text{if } x \in [0, \pi] \\ 2\pi - x, & \text{if } x \in [\pi, 2\pi] \\ -2\pi + x, & \text{if } x \in [2\pi, 3\pi] \text{ and so on.} \end{cases}$$

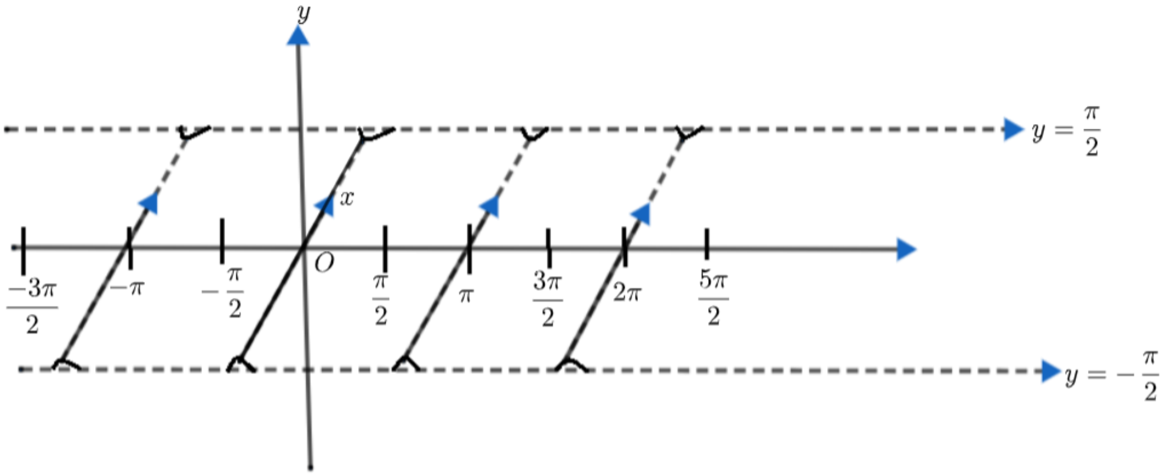
Next, we consider $y = \tan^{-1}(\tan x)$ which is periodic and has period π.

To draw this graph, we should draw the graph for one interval of length π and repeat the entire values of x.

We know $\tan^{-1}(\tan x) = \left\{ x; -\frac{\pi}{2} < x < \frac{\pi}{2} \right\}$. Thus, it has been defined for

$-\frac{\pi}{2} < x < \frac{\pi}{2}$ that has length π.

The graph is plotted as



Thus, the curve for $y = \tan^{-1}(\tan x)$, where y is not defined for $x \in (2n+1)\frac{\pi}{2}$.

The below result can be kept in mind.

$$\tan^{-1}(\tan x) = \begin{cases} -\pi - x, & \text{if } x \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right] \\ x, & \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x - \pi, & \text{if } x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ x - 2\pi, & \text{if } x \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \text{ and so on.} \end{cases}$$

Additional formulas

a. $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

b. $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

c. $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

d. $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$

e. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$, if $x>0, y>0, z>0 \,\&\, xy+yz+zx<1$

f. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$ when $x+y+z=xyz$

g. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$ when $xy+yz+zx=1$

h. $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\dfrac{3\pi}{2}$; $x=y=z=1$

i. $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$; $x=y=z=-1$

j. $\tan^{-1}1+\tan^{-1}2+2\tan^{-1}3=\tan^{-1}1+\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}=\dfrac{\pi}{2}$