FINAL JEE–MAIN EXAMINATION – APRIL, 2023

Held On Thursday 13th April, 2023

TIME : 09:00 AM to 12:00 PM

SECTION - A

1. $\int_0^\infty \frac{6}{e^{3x}+6e^{2x}+11e^x+6}dx$

(1) $\log_e\left(\frac{32}{27}\right)$ (2) $\log_e\left(\frac{256}{81}\right)$ (3) $\log_e\left(\frac{512}{81}\right)$ (4) $\log_e\left(\frac{64}{27}\right)$

Sol. 1

$$I=\int_0^\infty \frac{6}{(e^x+1)(e^x+2)(e^x+3)}dx$$

$$=6\int_0^\infty\left(\frac{\frac{1}{2}}{e^x+1}+\frac{-1}{e^x+2}+\frac{\frac{1}{2}}{e^x+3}\right)dx$$

$$=3\int_0^\infty\frac{e^{-x}}{1+e^{-x}}dx-6\int_0^\infty\frac{e^{-x}dx}{1+2e^{-x}}+3\int_0^\infty\frac{e^{-x}}{1+3e^{-x}}dx$$

$$=3\left[-\ln\left(1+e^{-x}\right)\right]_0^\infty+6\frac{1}{2}\left[\ln\left(1+2e^{-x}\right)\right]_0^\infty$$

$$-\frac{3}{3}\left[\ln\left(1+3e^{-x}\right)\right]_0^\infty$$

$$=3\ln 2-3\ln 3+\ln 4$$

$$=3\ln\frac{2}{3}+\ln 4$$

$$=\ln\frac{32}{27}$$

2. Among

(S1) : $\lim_{n\to\infty}\frac{1}{n^2}(2+4+6+.....+2n)=1$

(S2) : $\lim_{n\to\infty}\frac{1}{n^{16}}(1^{15}+2^{15}+3^{15}+.....+n^{15})=\frac{1}{16}$

(1) Only (S1) is true (2) Both (S1) and (S2) are true

(3) Both (S1) and (S2) are false (4) Only (S2) is true

Sol. 2

$$S_1:\lim_{n\to\infty}\frac{n(n+1)}{n^2}=1\Rightarrow \text{True}$$

$$S_2:\lim_{n\to\infty}\frac{1}{n^{16}}\left(\sum r^{15}\right)=\lim_{n\to\infty}\frac{1}{n}\sum\left(\frac{r}{n}\right)^{15}$$

$$=\int_0^1 x^{15}dx=\frac{1}{16}\Rightarrow \text{True}$$

3. The number of symmetric matrices of order 3, with all the entries from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, is

(1) 10^9 (2) 10^6 (3) 9^{10} (4) 6^{10}

Sol. **2**

$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$, $a, b, c, d, e, f \in \{0, 1, 2, \ldots .9\}$, Number of matrices = 10^6

4. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. If a vector $\vec{d}$ satisfies $\vec{d} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{d}.\vec{a} = 24$, then $|\vec{d}|^2$ is equal to -

(1) 323 (2) 423 (3) 413 (4) 313

Sol. **3**

$\vec{d} \times \vec{b} = \vec{c} \times \vec{b}$

$\Rightarrow (\vec{d} - \vec{c}) \times \vec{b} = 0$

$\Rightarrow \vec{d} = \vec{c} + \lambda\vec{b}$

Also $\vec{d}\vec{a} = 24$

$\Rightarrow (\vec{c} + \lambda\vec{b}) \cdot \vec{a} = 24$

$\lambda = \dfrac{24 - \vec{a} \cdot \vec{c}}{\vec{b} \cdot \vec{a}} = \dfrac{24 - 6}{9} = 2$

$\Rightarrow \vec{d} = \vec{c} + 2(\vec{b})$

$= 8\hat{i} - 5\hat{j} + 18\hat{k}$

$\Rightarrow |\vec{d}|^2 = 64 + 25 + 324 = 413$

5. A coin is biased so that the head is 3 times as likely to occur as tail. This coin is tossed until a head or three tails occur. If X denotes the number of tosses of the coin, then the mean of X is-

(1) $\dfrac{21}{16}$ (2) $\dfrac{15}{16}$ (3) $\dfrac{81}{64}$ (4) $\dfrac{37}{16}$

Sol. **1**

P(H) = $\dfrac{3}{4}$

P(T) = $\dfrac{1}{4}$

X	1	2	3
P(X)	$\frac{3}{4}$	$\frac{1}{4} \times \frac{3}{4}$	$\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^2 \times \frac{3}{4}$

$\text{Mean}\bar{X} = \dfrac{3}{4} + \dfrac{3}{8} + 3\left(\dfrac{1}{64} + \dfrac{3}{64}\right)$

$= \dfrac{3}{4} + \dfrac{3}{8} + \dfrac{3}{16}$

$= 3\left(\dfrac{7}{16}\right) = \dfrac{21}{16}$

6. $\max\limits_{0\le x\le\pi}\left\{x-2\sin x\cos x+\frac{1}{3}\sin 3x\right\}=$

(1) 0 (2) π (3) $\frac{5\pi+2+3\sqrt{3}}{6}$ (4) $\frac{\pi+2-3\sqrt{3}}{6}$

Sol. **3**

$$f(x)=x-\sin 2x+\frac{1}{3}\sin 3x$$

$$f'(x)=1-2\cos 2x+\cos 3x=0$$

$$x=\frac{5\pi}{6},\frac{\pi}{6}$$

$$\therefore f''(x)=4\sin 2x-3\sin 3x$$

$$f''\left(\frac{5\pi}{6}\right)<0$$

$\Rightarrow \left(\frac{5\pi}{6}\right)$ is point of maxima

$$f\left(\frac{5\pi}{6}\right)=\frac{5\pi}{6}+\frac{\sqrt{3}}{2}+\frac{1}{3}$$

7. The set of all a $\in$ R for which the equation x |x – 1| + |x + 2| + a = 0 has exactly one real root, is

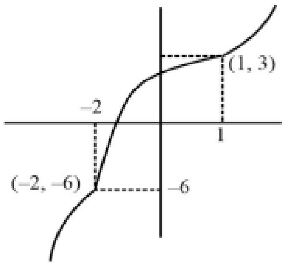
(1) $(-\infty,-3)$ (2) $(-\infty,\infty)$ (3) $(-6,\infty)$ (4) $(-6,-3)$

Sol. **2**

$$f(x)=x|x-1|+|x+2|$$

$$x|x-1|+|x+2|+a=0$$

$$x|x-1|+|x+2|=-a$$



All values are increasing.

8. Let PQ be a focal chord of the parabola $y^2 = 36x$ of length 100, making an acute angle with the positive x-axis. Let the ordinate of P be positive and M be the point on the line segment PQ such that PM:MQ = 3:1. Then which of the following points does <u>NOT</u> lie on the line passing through M and perpendicular to the line PQ?

(1) (3, 33) (2) (6, 29) (3) (– 6, 45) (4) (– 3, 43)

Sol. **4**

$$9\left(t+\frac{1}{t}\right)^2=100$$

$$t=3$$

$$\Rightarrow P(81,54)\ \&\ Q(1,-6)$$

$$M(21,9)$$

$$\Rightarrow \text{L is } (y-9)=\frac{-4}{3}(x-21)$$
$$3y-27=-4x+84$$
$$4x+3y=111$$

9. For the system of linear equations
$2x + 4y + 2az = b$
$x + 2y + 3z = 4$
$2x - 5y + 2z = 8$
which of the following is NOT correct?
(1) It has infinitely many solutions if a = 3, b = 8
(2) It has unique solution if a = b=8
(3) It has unique solution if a = b = 6
(4) It has infinitely many solutions if a = 3, b=6

Sol. **4**

$$\Delta=\begin{vmatrix}2 & 4 & 2a\\ 1 & 2 & 3\\ 2 & -5 & 2\end{vmatrix}=18(3-a)$$

$$\Delta_x=\begin{vmatrix}b & 4 & 2a\\ 4 & 2 & 3\\ 8 & -5 & 2\end{vmatrix}=(64+19b-72a)$$

For unique solution $\Delta = 0$
$$\Rightarrow a\neq 3 \text{ and } b\in R$$
For infinitely many solution :
$$\Delta=\Delta_x=\Delta_y=\Delta_z=0$$
$$\Rightarrow a=3 \quad \because \Delta=0$$
$$\text{and } b=8 \quad \because \Delta_x=0$$

10. Let $s_1, s_2, s_3, \ldots\ldots, s_{10}$ respectively be the sum to 12 terms of 10 A.P. s whose first terms are 1, 2, 3, ..., 10 and the common differences are 1, 3, 5, ..., 19 respectively. Then $\sum_{i=1}^{10} s_i$ is equal to

(1) 7260 (2) 7380 (3) 7220 (4) 7360

Sol. **1**

$$S_k=6(2k+(11)(2k-1))$$
$$S_k=6(2k+22k-11)$$
$$S_k=144k-66$$
$$\sum_{1}^{10}S_k=144\sum_{k=1}^{10}k-66\times 10$$
$$=144\times\frac{10\times 11}{2}-660$$
$$=7920-660$$
$$=7260$$

11. For the differentiable function f: R - {0} →R, let 3f(x)+2f$\left(\frac{1}{x}\right)=\frac{1}{x}-10$, then $\left|f(3)+f'\left(\frac{1}{4}\right)\right|$ is equal to

(1) 13 (2) $\frac{29}{5}$ (3) $\frac{33}{5}$ (4) 7

Sol. **1**

$$\left[3f(x)+2f\left(\frac{1}{x}\right)=\frac{1}{x}-10\right]\times 3$$

$$\left[2f(x)+3f\left(\frac{1}{x}\right)=x-10\right]\times 2$$

$$5f(x)=\frac{3}{x}-2x-10$$

$$f(x)=\frac{1}{5}\left(\frac{3}{x}-2x-10\right)$$

$$f'(x)=\frac{1}{5}\left(-\frac{3}{x^2}-2\right)$$

$$\left|f(3)+f'\left(\frac{1}{4}\right)\right|=\left|\frac{1}{5}(1-6-10)+\frac{1}{5}(-48-2)\right|$$

$$=|-3-10|=13$$

12. The negation of the statement $\left(\left(A\wedge(B\vee C)\right)\Rightarrow(A\vee B)\right)\Rightarrow A$ is

(1) equivalent to $B\vee\sim C$ (2) a fallacy
(3) equivalent to $\sim C$ (4) equivalent to $\sim A$

Sol. **4**

$$p:((A\wedge(B\vee C))\Rightarrow(A\vee B))\Rightarrow A$$

$$[\sim(A\wedge(B\vee C))\vee(A\vee B)]\Rightarrow A$$

$$[(A\wedge(B\vee C))\wedge\sim(A\vee B)]\vee A$$

$$(f\vee A)=A$$

$$\sim p\equiv\sim A$$

13. Let the tangent and normal at the point $\left(3\sqrt{3},1\right)$ on the ellipse $\frac{x^2}{36}+\frac{y^2}{4}=1$ meet the y-axis at the points A and B respectively. Let the circle C be drawn taking AB as a diameter and the line x = $2\sqrt{5}$ intersect C at the points P and Q. If the tangents at the points P and Q on the circle intersect at the point (α, β), then $\alpha^2-\beta^2$ is equal to

(1) $\frac{304}{5}$ (2) 60 (3) $\frac{314}{5}$ (4) 61

Sol. **1**

Given ellipse $\frac{x^2}{36}+\frac{y^2}{4}=1$

$$\frac{x}{4\sqrt{3}}+\frac{y}{4}=1$$

y = 4

$\frac{x}{4} - \frac{4}{4\sqrt{3}} = \frac{2}{\sqrt{3}}$

$y = -8$

$x^2 + y^2 + 4y - 32 = 0$

$hx + ky + 2(y + k) - 32 = 0$

$k = -2$

$hx + 2k - 32 = 0$

$hx = 36$

$\alpha = h = \frac{36}{2\sqrt{5}}$

$\beta = k = -2$

$\alpha^2 - \beta^2 = \frac{304}{5}$

14. The distance of the point (-1, 2, 3) from the plane $\vec{r}.\left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = 10$ parallel to the line of the shortest distance between the lines $\vec{r} = \left(\hat{i} - \hat{j}\right) + \lambda\left(2\hat{i} + \hat{k}\right)$ and $\vec{r} = \left(2\hat{i} - \hat{j}\right) + \mu\left(\hat{i} - \hat{j} + \hat{k}\right)$ is

(1) $2\sqrt{5}$ (2) $3\sqrt{5}$ (3) $3\sqrt{6}$ (4) $2\sqrt{6}$

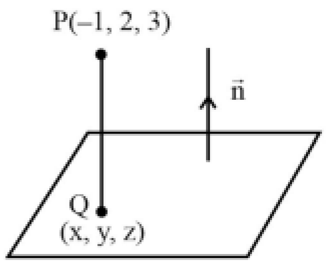
Sol. **4**

Let $L_1 : \vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{k})$

$L_2 : \vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$\vec{n} = \hat{i} - \hat{j} - 2\hat{k}$



Equation of line along shortest distance of L_1 and L_2

$\frac{x+1}{1} = \frac{y-2}{-1} = \frac{z-3}{-2} = r$

$\Rightarrow (x, y, z) \equiv (r - 1, 2 - r, 3 - 2r)$

$\Rightarrow (r - 1) - 2(2 - r) + 3(3 - 2r) = 10$

$\Rightarrow r = -2$

$\Rightarrow Q(x, y, z) \equiv (-3, 4, 7)$

$\Rightarrow PQ = \sqrt{4 + 4 + 16} = 2\sqrt{6}$

15. Let $B=\begin{bmatrix}1 & 3 & \alpha\\ 1 & 2 & 3\\ \alpha & \alpha & 4\end{bmatrix}$, $\alpha > 2$ be the adjoint of a matrix A and |A|=2. then

$\begin{bmatrix}\alpha & -2\alpha & \alpha\end{bmatrix} B \begin{bmatrix}\alpha\\ -2\alpha\\ \alpha\end{bmatrix}$ is equal to

(1) 16 (2) 32 (3) 0 (4) –16

Sol. **4**

$$\text{Given, } B=\begin{bmatrix}1 & 3 & \alpha\\ 1 & 2 & 3\\ \alpha & \alpha & 4\end{bmatrix}$$

$$|B|=4$$

$$1(8-3\alpha)-3(4-3\alpha)+\alpha(\alpha-2\alpha)=4$$

$$-\alpha^2+6\alpha-8=0$$

$$\alpha=2,4$$

Given $\alpha > 2$

So, $\alpha = 2$ is rejected

$$\begin{bmatrix}4 & -8 & 4\end{bmatrix}\begin{bmatrix}1 & 3 & 4\\ 1 & 2 & 3\\ 4 & 4 & 4\end{bmatrix}\begin{bmatrix}4\\ -8\\ 4\end{bmatrix}=[-16]_{1\times 1}$$

16. For $x \in R$, two real valued functions f(x) and g(x) are such that, g(x)= $\sqrt{x}+1$ and fog(x)=x+3 $-\sqrt{x}$. Then f(0) is equal to

(1) 5 (2) 0 (3) –3 (4) 1

Sol. **1**

$$g(x)=\sqrt{x}+1$$

$$fog(x)=x+3-\sqrt{x}$$

$$=(\sqrt{x}+1)^2-3(\sqrt{x}+1)+5$$

$$=g^2(x)-3g(x)+5$$

$$\Rightarrow f(x)=x^2-3x+5$$

$$\therefore f(0)=5$$

But, if we consider the domain of the composite function fog(x) then in that case f(0) will be not defined as g(x) cannot be equal to zero.

17. Let the equation of plane passing through the line of intersection of the planes x+2y+az=2 and x-y+z=3 be 5x – 11y + bz = 6a – 1. For $c \in Z$, if the distance of this plane from the point (a, –c, c) is $\frac{2}{\sqrt{a}}$, then $\frac{a+b}{c}$ is equal to

(1) – 4 (2) 2 (3) – 2 (4) 4

Sol. **1**

$$(x+2y+az-2)+\lambda(x-y+z-3)=0$$

$$\frac{1+\lambda}{5}=\frac{2-\lambda}{-11}=\frac{a+\lambda}{b}=\frac{2+3\lambda}{6a-1}$$

$$\lambda=-\frac{7}{2}, a=3, b=1$$

$$\frac{2}{\sqrt{a}}=\left|\frac{5a+11c+bc-6a+1}{\sqrt{25+121+1}}\right|$$

$$c=-1$$

$$\therefore \frac{a+b}{c}=\frac{3+1}{-1}=-4$$

18. Fractional part of the number is $\frac{4^{2022}}{15}$ equal to

(1) $\frac{4}{15}$ (2) $\frac{8}{15}$ (3) $\frac{1}{15}$ (4) $\frac{14}{15}$

Sol. **3**

$$\left\{\frac{4^{2022}}{15}\right\}=\left\{\frac{2^{4044}}{15}\right\}=\left\{\frac{(1+15)^{1011}}{15}\right\}=\frac{1}{15}$$

19. Let y = y_1 (x) and y = y_2(x) be the solution curves of the differential equation $\frac{dy}{dx}$ = y+7 with initial conditions y_1(0) = 0 and y_2(0)=1 respectively. Then the curves y = y_1(x) and y = y_2(x) intersect at

(1) no point
(2) infinite number of points
(3) one point
(4) two points

Sol. **1**

$$\frac{dy}{dx}=y+7\Rightarrow\frac{dy}{dx}-y=7$$

$$I.F.=e^{-x}$$

$$ye^{-x}=\int 7e^{-x}dx$$

$$\Rightarrow ye^{-x}=-7e^{-x}+c$$

$$\Rightarrow y=-7+ce^{x}$$

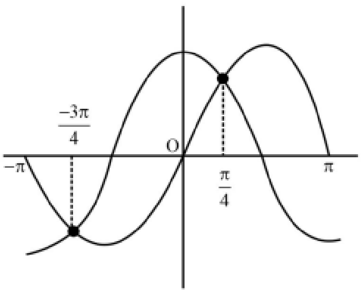
$$-7+7e^{x}=-7+8e^{x}\Rightarrow e^{x}=0.$$

No solution

20. The area of the region enclosed by the curve f(x)= max{sin x, cosx}, $-\pi \le x \le \pi$ and the x-axis is

(1) $2\sqrt{2}\left(\sqrt{2}+1\right)$
(2) $4\left(\sqrt{2}\right)$
(3) 4
(4) $2\left(\sqrt{2}+1\right)$

Sol. **3**



Area =

$$\left|\int_{-\pi}^{\frac{-3\pi}{4}} \sin x dx\right| + \left|\int_{\frac{-3\pi}{4}}^{\frac{-\pi}{2}} \cos x dx\right| + \int_{\frac{-\pi}{2}}^{\frac{\pi}{4}} \cos x dx + \int_{\frac{\pi}{4}}^{\pi} \sin x dx = 4$$

SECTION - B

21. The sum to 20 terms of the series $2.2^2 - 3^2 + 2.4^2 - 5^2 + 2.6^2 -$ .... is equal to ________.

Sol. **1310**

$$\left(2^2 - 3^2 + 4^2 - 5^2 + 20\text{terms}\right) + \left(2^2 + 4^2 + \ldots + 10\text{terms}\right)$$

$$-(2+3+4+5+\ldots+11) + 4\left[1+2^2+\ldots 10^2\right]$$

$$-\left[\frac{21\times 22}{2} - 1\right] + 4\times\frac{10\times 11\times 21}{6}$$

$$= 1 - 231 + 14\times 11\times 10$$

$$= 1540 + 1 - 231$$

$$= 1310$$

22. Let the mean of the data

x	1	3	5	7	9
Frequency (f)	4	24	28	α	8

be 5. If m and σ^2 are respectively the mean deviation about the mean and the variance of the data, then $\frac{3\alpha}{m+\sigma^2}$ is equal to ________.

Sol. **8**

$$5 = \bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{4+72+140+7\alpha+72}{64+\alpha}$$

$$\Rightarrow 320 + 5\alpha = 288 + 7\alpha \Rightarrow 2\alpha = 32 \Rightarrow \alpha = 16$$

$$\text{M.D.}(\bar{x}) = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i} \text{ where} \sum f_i = 64 + 16 = 80$$

$$\text{M.D. }(\bar{x}) = \frac{4\times 4 + 24\times 2 + 28\times 0 + 16\times 2 + 8\times 4}{80} = \frac{8}{5}$$

$$\text{Variance} = \frac{\sum f_i\left(x_i-\bar{x}\right)^2}{\sum f_i}$$

$$= \frac{4\times16+24\times4+0+16\times4+8\times16}{80} = \frac{352}{80}$$

$$\therefore \frac{3\alpha}{m+\sigma^2} = \frac{3\times16}{\frac{128}{80}+\frac{352}{80}} = 8$$

23. Let α be the constant term in the binomial expansion of $\left(\sqrt{x}-\frac{6}{x^{\frac{3}{2}}}\right)^n$, $n\le15$. If the sum of the coefficients of the remaining terms in the expansion is 649 and the coefficient of x^{-n} is λα, then λ is equal to ________.

Sol. **36**

$$T_{k+1} = {}^nC_k(x)^{\frac{n-k}{2}}(-6)^k(x)^{\frac{-3}{2}k}$$

$$\frac{n-k}{2}-\frac{3}{2}k=0$$

$$n-4k=0$$

$$(-5)^n-\left({}_nC_{\frac{n}{4}}(-6)^{\frac{n}{4}}\right)=649$$

By observation $(625 + 24 = 649)$, we get n = 4

$\because n = 4 \,\&\, k = 1$

Required is coefficient of

$$x^{-4} \text{ is } \left(\sqrt{4}-\frac{6}{x^{\frac{3}{2}}}\right)^4$$

$${}^4C_1(-6)^3$$

By calculating we will get $\lambda = 36$

24. Let $\omega = z\bar{z}+k_1z+k_2iz+\lambda(1+i)$, $k_1,k_2\in R$. Let Re(ω) = 0 be the circle C of radius 1 in the first quadrant touching the line y = 1 and the y-axis. If the curve Im(ω) = 0 intersects C at A and B, then $30(AB)^2$ is equal to ______.

Sol. **24**

$$\omega = z\bar{z}+k_1z+k_2iz+\lambda(1+i)$$

$$\text{Re}(\omega) = x^2+y^2+k_1x-k_2y+\lambda=0$$

$$\text{Centre} \equiv \left(\frac{-k_1}{2},\frac{k_2}{2}\right)\equiv(1,2)$$

$$\Rightarrow k_1=-2, k_2=4$$

$$\text{radius}=1\Rightarrow\lambda=4$$

$$\text{Im}=k_1y+k_2x+\lambda=0$$

$$\therefore 2x-y+2=0$$

$d = \frac{2}{\sqrt{5}}$

$\frac{1^2}{4} = 1 - \frac{4}{5} = \frac{1}{5}$

$\therefore 301^2 = 24$

25. Let $\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - 3\hat{j} + 3\hat{k}$. If $\vec{b}$ is a vector such that $\vec{a} = \vec{b} \times \vec{c}$ and $|\vec{b}|^2 = 50$, then $\left|72 - |\vec{b} + \vec{c}|^2\right|$ is equal to ______.

Sol. **66**

$|\vec{a}| = \sqrt{11}, |\vec{c}| = \sqrt{22}$

$|\vec{a}| = |\vec{b} \times \vec{c}| = |\vec{b}||\vec{c}| \sin\theta$

$\sqrt{11} = \sqrt{50}\sqrt{22} \sin\theta$

$\Rightarrow \sin\theta = \frac{1}{10}$

$|\vec{b} + \vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b} \cdot \vec{c}$

$= |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\theta$

$= 50 + 22 + 2 \times \sqrt{50} \times \sqrt{22} \times \frac{\sqrt{99}}{10}$

$= 72 + 66$

$\left|72 - |\vec{b} + \vec{c}|^2\right| = 66$

26. Let m_1, and m_2 be the slopes of the tangents drawn from the point P(4,1) to the hyperbola H: $\frac{y^2}{25} - \frac{x^2}{16} = 1$. If Q is the point from which the tangents drawn to H have slopes $|m_1|$ and $|m_2|$ and they make positive intercepts α and β on the x-axis, then $\frac{(PQ)^2}{\alpha\beta}$ is equal to ______.

Sol. **8**

Equation of tangent to the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

$y = mx \pm \sqrt{a^2 - b^2m^2}$

passing through (4, 1)

$1 = 4m \pm \sqrt{25 - 16m^2} \Rightarrow 4m^2 - m - 3 = 0$

$\Rightarrow m = 1, \frac{-3}{4}$

Equation of tangent with positive slopes $1 \& \frac{3}{4}$

$\left.\begin{aligned} 4y &= 3x - 16 \\ y &= x - 3 \end{aligned}\right\}$ with positive intercept on x-axis.

$\alpha = \frac{16}{3}, \beta = 3$

Intersection points:

Q : (–4, –7)

P : (4, 1)

$PQ^2 = 128$

$\frac{PQ^2}{\alpha\beta} = \frac{128}{16} = 8$

27. Let the image of the point $\left(\frac{5}{3}, \frac{5}{3}, \frac{8}{3}\right)$ in the plane x – 2y + z – 2 = 0 be P. If the distance of the point Q(6, –2, α), α > 0, from P is 13, then α is equal to _______.

Sol. **15**

Image of point $\left(\frac{5}{3}, \frac{5}{3}, \frac{8}{3}\right)$

$$\frac{x-\frac{5}{3}}{1} = \frac{y-\frac{5}{3}}{-2} = \frac{z-\frac{8}{3}}{1} = \frac{-2\left(1\times\frac{5}{3}+(-2)\times\frac{8}{3}+1\times\frac{8}{3}-2\right)}{1^2+2^2+1^2}$$

$$= \frac{1}{3}$$

$\therefore x = 2, y = 1, z = 3$

$13^2 = (6-2)^2 + (-2-1)^2 + (\alpha-3)^2$

$\Rightarrow (\alpha-3)^2 = 144 \Rightarrow \alpha = 15 (\because \alpha > 0)$

28. Let for $x \in R$, $S_0(x) = x, S_k(x) = C_k x + k\int_0^x S_{k-1}(t)dt$ where $C_0 = 1$, $C_k = 1 - \int_0^1 S_{k-1}(x)dx$, $k = 1, 2, 3, \ldots$ Then $S_2(3) + 6C_3$ is equal to ___________.

Sol. **18**

Given, $S_k(x) = C_k x + k\int_0^x S_{k-1}(t)dt$

put $k = 2$ and x = 3

$S_2(3) = C_2(3) + 2\int_0^3 S_1(t)dt$ ……(1)

Also, $S_1(x) = C_1(x) + \int_0^x S_0(t)dt$

$= C_1 x + \frac{x^2}{2}$

$S_2(3) = 3C_2 + 2\int_0^3 \left(C_1 t + \frac{t^2}{2}\right)dt$

$= 3C_2 + 9C_1 + 9$

Also,

$C_1 = 1 - \int_0^1 S_0(x)dx = \frac{1}{2}$

$$C_2 = 1 - \int_0^1 S_1(x)dx = 0$$

$$C_3 = 1 - \int_0^1 S_2(x)dx$$

$$= 1 - \int_0^1 \left(C_2 x + C_1 x^2 + \frac{x^3}{3} \right) dx = \frac{3}{4}$$

$$S_2(x) = C_2 x + 2\int_0^x S_1(t)dt$$

$$= C_2 x + C_1 x^2 + \frac{x^3}{3}$$

$$\Rightarrow S_2(3) + 6C_3 = 6C_3 + 3C_2 + 9C_1 + 9 = 18$$

29. If $S = \left\{ x \in R : \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}} \right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}} \right) = \frac{\pi}{4} \right\}$, then $\sum_{x \in S} \left(\sin\left((x^2+x+5)\frac{\pi}{2} \right) - \cos\left((x^2+x+5)\pi \right) \right)$ is equal to __________.

Sol. **4**

$$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}} \right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}} \right) = \frac{\pi}{4}$$

$$\because \frac{t}{\sqrt{t^2+1}} \in (-1,1)$$

$$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}} \right) = \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}} \right) + \frac{\pi}{4}$$

$$\frac{(x+1)}{\sqrt{(x+1)^2+1}} = \left(\frac{1}{\sqrt{2}} \right) \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{x^2+1}} \right) \right) + \frac{1}{\sqrt{2}} \left(\frac{x}{\sqrt{x^2+1}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}} \right)$$

$$\frac{(x+1)}{\sqrt{(x+1)^2+1}} = \frac{1}{\sqrt{2}} \left(\frac{1+x}{\sqrt{x^2+1}} \right)$$

After solving this equation, we get

x = - 1 or x = 0

$S = \{-1, 0\}$

$$\sum_{x\in R}\left(\sin\left(\left(x^2+x+5\right)\frac{\pi}{2}\right)-\cos\left(\left(x^2+x+5\right)\pi\right)\right)$$

$$=\left[\sin\left(\frac{5\pi}{2}\right)-\cos(5\pi)\right]+\left[\sin\left(\frac{5\pi}{2}\right)-\cos(5\pi)\right]=4$$

30. The number of seven digit positive integers formed using the digits 1, 2, 3 and 4 only and sum of the digits equal to 12 is ________.

Sol. **413**

$x_1+x_2+x_3+x_4+x_5+x_6+x_7=12, x_i\in\{1,2,3,4\}$

No. of solutions $={}^{5+7-1}C_{7-1}-\frac{7!}{6!}-\frac{7!}{5!}=413$