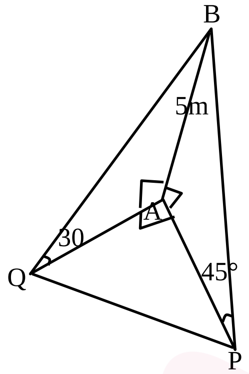
FINAL JEE–MAIN EXAMINATION – APRIL, 2023
Held On Tuesday 11th April, 2023
TIME : 03:00 PM to 06:00 PM

SECTION - A

1. The angle of elevation of the top P of a tower from the feet of one person standing due South of the tower is 45° and from the feet of another person standing due west of the tower is 30°. If the height of the tower is 5 meters, then the distance (in meters) between the two persons is equal to

(1) 10 (2) $5\sqrt{5}$ (3) $\frac{5}{2}\sqrt{5}$ (4) 5

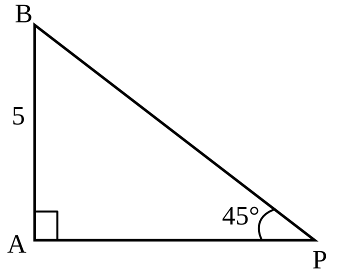
Sol. **(1)**



Tower AB = 5 m

∠APB = 45°

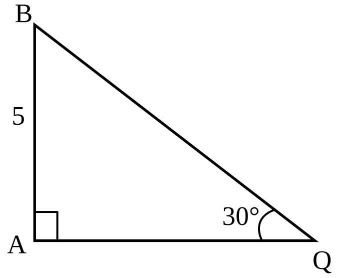
∠PAB = 90°



$$\tan 45° = \frac{AB}{AP}$$

$$1 = \frac{AB}{AP}$$

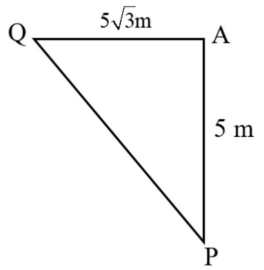
$$\boxed{AP = 5m}$$



$$\tan 30° = \frac{AB}{AQ}$$

$$\frac{1}{1\sqrt{3}} = \frac{5}{AQ}$$

$\boxed{AQ = 5\sqrt{3}}$



$AP^2 + AQ^2 = PQ^2$

$PQ^2 = 5^2 + \left(5\sqrt{3}\right)^2$

$PQ^2 = 25 + 75 = 100$

$\boxed{PQ = 10cm}$

Option (A) 10 cm correct.

2. Let a, b, c and d be positive real numbers such that a + b + c + d = 11. If the maximum value of $a^5 b^3 c^2 d$ is 3750β, then the value of β is

(1) 55 (2) 108 (3) 90 (4) 110

Sol. (3)

Given a + b + c + d = 11 (a, b, c, d > 0}

$(a^5b^3c^2d)$max. = ?

Let assume Numbers –

$\frac{a}{5}, \frac{a}{5}, \frac{a}{5}, \frac{a}{5}, \frac{a}{5}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2},$

We know A.M. ≥ G.M.

$$\frac{\frac{a}{5}+\frac{a}{5}+\frac{a}{5}+\frac{a}{5}+\frac{a}{5}+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}+d}{11} \geq \left(\frac{a^5b^3c^2d}{5^5.3^3.2^2.1}\right)^{\frac{1}{11}}$$

$$\frac{11}{11} \geq \left(\frac{a^5b^3c^2d}{5^5.3^3.2^2.1}\right)^{\frac{1}{11}}$$

$a^5.b^3.c^2.d \leq 5^5.3^3.2^2,$

$\max(a^5b^3c^2d) = 5^5.3^3.2^2 = 337500$

$= 90 \times 3750 = \beta \times 3750$

$\boxed{\beta = 90}$

Option (C) 90 correct

3. If $f : R \to R$ be a continuous function satisfying $\int_0^{\frac{\pi}{2}} f(\sin 2x)\sin x dx + \alpha \int_0^{\frac{\pi}{4}} f(\cos 2x)\cos x dx = 0$, then the value of α is

(1) $-\sqrt{3}$ (2) $\sqrt{3}$ (3) $-\sqrt{2}$ (4) $\sqrt{2}$

Sol. **(3)**

$F : R \to R$

$$\Rightarrow \int_0^{\frac{\pi}{2}} F(\sin 2x)\sin x dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x).\cos x dx = 0$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} F(\sin 2x)\sin x dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\sin 2x).\sin x dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x).\cos x dx = 0$$

$$\int_0^a F(x) dx = \int_0^a F(a-x) dx$$

Let $x = t + \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\sin\left(\frac{\pi}{4} - x\right)dx + \int_0^{\frac{\pi}{4}} F(\cos 2t)\sin\left(t + \frac{\pi}{4}\right) + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x)\cos x dx = 0$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\left\{\sin\left(\frac{\pi}{4} - x\right) + \sin\left(x + \frac{\pi}{4}\right) + \alpha\cos x = 0\right.$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\left\{\left(\sqrt{2} + \alpha\right)\cos x\right\} dx = 0$$

$$\left(\sqrt{2} + \alpha\right)\int_0^{\frac{\pi}{4}} F(\cos 2x)\cos x\, dx = 0$$

$\because$ in interval $\left(0, \frac{\pi}{4}\right) \Rightarrow F(\cos 2x)$ & $\cos x$ is NOT Zero.

$\therefore \sqrt{2} + \alpha = 0$

$\boxed{\alpha = -\sqrt{2}}$

4. Let f and g be two functions defined by $f(x) = \begin{cases} x+1, & x < 0 \\ |x-1|, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x+1, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Then (gof) (x) is

(1) continuous everywhere but not differentiable at x = 1

(2) continuous everywhere but not differentiable exactly at one point

(3) differentiable everywhere

(4) not continuous at x = –1

Sol. **(2)**

$$f(x) = \begin{cases} x+1, x < 0 \\ 1-x, 0 \leq x < 1 \\ x-1, 1 \leq x \end{cases}$$

$$g(x) = \begin{cases} x+1, x < 0 \\ 1, x \geq 0 \end{cases}$$

$$g(f(x)) = \begin{cases} x+2, x<-1 \\ 1, x \geq -1 \end{cases}$$

$\therefore$ g(f (x)) is continuous everywhere

g(f (x)) is not differentiable at x = – 1

Differentiable everywhere else

5. If the radius of the largest circle with centre (2, 0) inscribed in the ellipse $x^2 + 4y^2 = 36$ is r, then $12r^2$ is equal to

(1) 69 (2) 72 (3) 115 (4) 92

Sol. **(4)**

C(2,0)

Ellipse $x^2 + 4y^2 = 36$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

Equation of Normal at P(6cosθ, 3sinθ) is (6secθ)x – (3cosecθ)y = 27

It passes through (2,0)

$$\Rightarrow \sec\theta = \frac{27}{12} = \frac{9}{4}$$

$$\cos\theta\ \frac{4}{9},\ \sin\theta = \frac{\sqrt{65}}{9}$$

$$P\left(\frac{8}{3}, \frac{\sqrt{65}}{3}\right)$$

$$\overset{\gamma}{\overline{P\left(\frac{8}{3}, \frac{\sqrt{65}}{3}\right)\quad c(2,0)}}$$

$$\gamma = \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(\frac{\sqrt{65}}{3}\right)^2} = \frac{\sqrt{69}}{3}$$

$$\text{Value of } 12\gamma^2 = \left(\frac{\sqrt{69}}{3}\right)^2 \times 12$$

$$\Rightarrow \frac{12 \times 69}{9} = 92$$

6. Let the mean of 6 observations 1, 2, 4, 5 x and y 5 and their variance be 10. Then their mean deviation about the mean is equal to

(1) $\frac{7}{3}$ (2) $\frac{10}{3}$ (3) $\frac{8}{3}$ (4) 3

Sol. **(3)**

Mean of 1, 2, 4, 5, x, y is 5

and variance is 10

$$\Rightarrow \text{mean} \Rightarrow \frac{12 + x + y}{6} = 5$$

$12 + x + y = 30$

$x + y = 18$

and by variance $\frac{x^2 + y^2 + 46}{6} - 5^2 = 10$

$x^2 + y^2 = 164$

$x = 8 \quad y = 10$

mean daviation = $\frac{|x - \bar{x}|}{6}$

$\Rightarrow \frac{4+3+1+0+3+5}{6} = \frac{16}{6} = \frac{8}{3}$

7. Let A = {1, 3, 4, 6, 9} and B = {2, 4, 5, 8, 10}. Let R be a relation defined on A×B such that R ={($(a_1, b_1), (a_2, b_2)$): $a_1 \le b_2$ and $b_1 \le a_2$}. Then the number of elements in the set R is

(1) 52 (2) 160 (3) 26 (4) 180

Sol. **(2)**

Let $a_1 = 1 \Rightarrow 5$ choices of b_2

$a_1 = 3 \Rightarrow 4$ choices of b_2

$a_1 = 4 \Rightarrow 4$ choices of b_2

$a_1 = 6 \Rightarrow 2$ choices of b_2

$a_1 = 9 \Rightarrow 1$ choices of b_2

For (a_1, b_2) 16 ways .

Similarly, $b_1 = 2 \Rightarrow 4$ choices of a_2

$b_1 = 4 \Rightarrow 3$ choices of a_2

$b_1 = 5 \Rightarrow 2$ choices of a_2

$b_1 = 8 \Rightarrow 1$ choices of a_2

Required elements in R = 160

8. Let P be the plane passing through the points (5, 3, 0), (13, 3, –2) and (1, 6, 2). For $\alpha \in N$, if the distances of the points A(3, 4, α) and B(2, α, a) from the plane P are 2 and 3 respectively, then the positive value of a is

(1) 5 (2) 6 (3) 4 (4) 3

Sol. **(3)**

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 0 & -2 \\ 4 & -3 & -2 \end{vmatrix} = \hat{i}(-6) + 8\hat{j} - 24\hat{k}$$

Normal of the plane $= 3\hat{i} - 4\hat{j} + 12\hat{k}$

Plane : $3x - 4y + 12z = 3$

Distance from A(3,4, α)

$$\left|\frac{9 - 16 + 12\alpha - 3}{13}\right| = 2$$

$\alpha = 3$

$\alpha = -8$ (rejected)

Distance from B(2,3,a)

$$\left|\frac{6 - 12 + 12a - 3}{13}\right| = 3$$

$a = 4$

9. If the letters of the word MATHS are permuted and all possible words so formed are arranged as in a dictionary with serial number, then the serial number of the word THAMS is

(1) 102 (2) 103 (3) 101 (4) 104

Sol. **(2)**

5	2	1	3	4
T	H	A	M	S
4	1	0	0	0
4!	3!	2!	1!	0!

$\Rightarrow 4 \times 4! + 3! \times 1 + 0 + 0 + 0$

$\Rightarrow 96 + 6 = 102$

Ran k THAMS = 102 + 1 = 103

10. If four distinct points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar, then $[\vec{a}\vec{b}\vec{c}]$ is equal to

(1) $[\vec{d}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{d}\ \vec{a}] + [\vec{c}\ \vec{d}\ \vec{b}]$ (2) $[\vec{d}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{b}\ \vec{c}]$

(3) $[\vec{a}\ \vec{d}\ \vec{b}] + [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{d}\ \vec{b}\ \vec{c}]$ (4) $[\vec{b}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{a}\ \vec{c}] + [\vec{d}\ \vec{b}\ \vec{a}]$

Sol. **(1)**

$\vec{a}, \vec{b}, \vec{c}, \vec{d} \rightarrow$ coplanar

$[\vec{a}\ \vec{b}\ \vec{c}] = ?$

$\vec{b} - \vec{a}, \vec{c} - \vec{b}, \vec{d} - \vec{c} \rightarrow$ coplanar

$[\vec{b} - \vec{a}\ \ \vec{c} - \vec{b},\ \vec{d} - \vec{c}] = 0$

$\Rightarrow (\vec{b} - \vec{a}).((\vec{c} - \vec{b}) \times (\vec{d} - \vec{c})) = 0$

$(\vec{b} - \vec{a}).(\vec{c} \times \vec{b} - \vec{c} \times \vec{a} - \vec{a} \times \vec{d}) = 0$

$[bcd] - [bca] - [bad] - [acd] = 0$

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{d}\ \vec{a}] + [\vec{c}\ \vec{d}\ \vec{b}]$

11. The sum of the coefficients of three consecutive terms in the binomial expansion of $(1 + x)^{n+2}$, which are in the ratio 1 : 3 : 5, is equal to

(1) 63 (2) 92 (3) 25 (4) 41

Sol. **(1)**

$^{n+2}C_{r-1} : {}^{n+2}C_r : {}^{n+2}C_{r+1} :: 1 : 3 : 5$

$$\frac{(n+2)!}{(r-1)!(n-r+3)!} \times \frac{r!(n+2-r)!}{(n+2)!} = \frac{1}{3}$$

$$\frac{r}{(n-r+3)} = \frac{1}{3} \Rightarrow n-r+3 = 3r$$

$\boxed{n = 4r-3} - 0$

$$\frac{(n+1)!}{r!(n+2-r)!} \times \frac{(r+1)!(n-r+1)!}{(n+2)!} = \frac{3}{5}$$

$$\frac{r+1}{n+2-r} = \frac{3}{5}$$

8r–1= 3n ……..(2)

By equation 1 and 2

$\frac{8r-1}{3} = 4r-3$ $\qquad$ $n = 4r-3$

$\boxed{r=2}$ $\qquad$ $n = 4 \times 2 - 3$

$\boxed{n=5}$

Sum : ${}^7C_1 + {}^7C_2 + {}^7C_3 = 7 + 21 + 35 = 63$

12. Let y = y (x) be the solution of the differential equation $\frac{dy}{dx} + \frac{5}{x(x^5+1)} y = \frac{(x^5+1)^2}{x^7}$, x > 0. If y(1) = 2, then y(2) is equal to

(1) $\frac{693}{128}$ (2) $\frac{637}{128}$ (3) $\frac{697}{128}$ (4) $\frac{679}{128}$

Sol. **(1)**

I.F = $= e^{\int \frac{5dx}{x(x^5+1)}} = e^{e^{\int \frac{5x^{-6}dx}{(x^{-5}+1)}}}$

Put, $1 + x^{-5} = t \Rightarrow -5x^{-6}\,dx = dt$

$$\Rightarrow e^{\int \frac{-dt}{1}} = \frac{1}{t} = \frac{x^5}{1+x^5}$$

$$y \cdot \frac{x^5}{1+x^5} = \int \frac{x^5}{(1+x^5)} \times \frac{(1+x^5)^2}{x^7} dx$$

$$= \int x^3 dx + \int x^{-2} dx$$

$$y \cdot \frac{x^5}{1+x^5} = \frac{x^4}{4} - \frac{1}{x} + c$$

Given than: x = 1 $\Rightarrow$ y = 2

$$2 \cdot \frac{1}{2} = \frac{1}{4} - 1 + c$$

$$c = \frac{7}{4}$$

$$y \cdot \frac{x^5}{1+x^5} = \frac{x^4}{4} - \frac{1}{x} + \frac{7}{4}$$

Now put, x = 2

$$y \cdot \left(\frac{32}{33}\right) = \frac{21}{4}$$

$$y = \frac{693}{128}$$

13. The converse of $((\sim p) \wedge q) \Rightarrow r$ is

(1) $(pv(\sim q)) \Rightarrow (\sim r)$ (2) $((\sim p)vq) \Rightarrow r$ (3) $(\sim r) \Rightarrow ((\sim p) \wedge q)$ (4) $(\sim r) \Rightarrow p \wedge q$

Sol. **(1)**

$((-P) \wedge 2) \Rightarrow r$

Converse ….

$\sim((\sim P) \wedge q) \Rightarrow (\sim r)$

$(P \vee (\sim q)) \Rightarrow (\sim r)$

14. If the 1011[th] term from the end in the binominal expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^{2022}$ is 1024 times 1011[th] term from the beginning, the |x| is equal to

(1) 8 (2) 12 (3) 10 (4) 15

Sol. **(3)– Bouns**

T_{1011} from beginning = T_{1010+1}

$$= {}^{2022}C_{1010} \left(\frac{4x}{5}\right)^{1012} \left(\frac{-5}{2x}\right)^{1010}$$

T_{1011} from end

$$= {}^{2022}C_{1010} \left(\frac{-5}{2x}\right)^{1012} \left(\frac{4x}{5}\right)^{1010}$$

Given: $= {}^{2022}C_{1010} \left(\frac{-5}{2x}\right)^{1012} \left(\frac{4x}{5}\right)^{1010}$

$$= 2^{10} \cdot {}^{2022}C_{1010} \left(\frac{-5}{2x}\right)^{1010} \left(\frac{4x}{5}\right)^{1012}$$

$$\left(\frac{-5}{2x}\right)^2 = 2^{10} \left(\frac{4x}{5}\right)^2$$

$$x^4 = \frac{5^4}{2^{16}}$$

$$|x| = \frac{5}{16}$$

15. If the system of linear equations

$7x + 11y + \alpha z = 13$

$5x + 4y + 7z = \beta$

$175x + 194y + 57z = 361$

has infinitely many solutions, then $\alpha + B + 2$ is equal to :

(1) 3 (2) 6 (3) 5 (4) 4

Sol. **(4)**

$7x + 11y + \alpha z = 13$

$5x + 4y + 7z = \beta$

$175x + 194y + 57z = 361$

4sc condition of Infinite Many solution

$\boxed{\Delta = 0}$ & $\Delta x, \Delta y, \Delta z = 0$ check.

After solving we get $\boxed{\alpha + 13 + 2 = 4}$

16. Let the line passing through the point P (2, –1, 2) and Q (5, 3, 4) meet the plane x – y +z = 4 at the point T. Then the distance of the point R from the plane x + 2y + 3z +2 = 0 measured parallel to the line $\frac{x-7}{2} = \frac{y+3}{2} = \frac{z-2}{1}$ is equal to

(1) 3 (2) $\sqrt{61}$ (3) $\sqrt{31}$ (4) $\sqrt{189}$

Sol. **(1)**

$\text{Line}: \frac{x-5}{3} = \frac{y-3}{4} = \frac{z-4}{2} = \lambda$

$R(3\lambda + 5, 4\lambda + 3, 2\lambda + 4)$

$\therefore 3\lambda + 5 - 4\lambda - 3 + 2\lambda + 4 = 4$

$\lambda + 6 = 4 \qquad \therefore \lambda = -2$

$\therefore R \equiv (-1, -5, 0)$

$\text{Line}: \frac{x+1}{2} = \frac{y+5}{2} = \frac{z-0}{1} = \mu$

Point T = $(2\mu - 1, 2\mu - 5, \mu)$

It lies on plane

$2\mu - 1 + 2(2\mu - 5) + 3\mu + 2 = 0$

$\mu = 1$

$\therefore T = (1, -3, 1)$

$\therefore RT = 3$

17. Let the function f : [0, 2] $\rightarrow$ R be defined as

$$f(x) = \begin{cases} e^{\min\{x^2, x-[x]\}}, & x \in [0,1) \\ e^{[x - \log_e x]}, & x \in [1,2) \end{cases}$$

where [t] denotes the greatest integer less than or equal to t. Then the value of the integral $\int_0^2 xf(x)dx$ is

(1) $(e-1)\left(e^2 + \frac{1}{2}\right)$ (2) $1 + \frac{3e}{2}$ (3) $2e - \frac{1}{2}$ (4) $2e - 1$

Sol. **(3)**

$F[0,2] \to R$

$$F(x)=\begin{cases}\min\{x^2,\{x\}\}; x\in[0,1)\\ [x-\log_e x]=1; x\in[1,2)\end{cases}$$

$$F(x)=\begin{cases}e^{x^2} : x\in[0,1)\\ e \quad x\in[1,2)\end{cases}$$

$$\int_0^2 xf(x)dx = \int_0^1 x.e^{x^2}dx + \int_1^2 x.e\,dx$$

$$= \frac{1}{2}(e-1)+\frac{1}{2}(4-1)e$$

$$\Rightarrow 2e-\frac{1}{2}$$

18. For $a\in C$, let A = $\{z\in C: \text{Re}(a+\bar{z})>\text{Im}(\bar{a}+z)\}$ and B = $\{z\in C: \text{Re}(a+\bar{z})<\text{Im}(\bar{a}+z)\}$. The among the two statements:

(S1) : If Re (a), Im (a) >0, then the set A contains all the real numbers

(S2) : If Re (a), Im (a) < 0, then the set B contains all the real numbers,

(1) only (S1) is true (2) both are false (3) only (S2) is true (4) both are true

Sol. **(2)**

Let $a = x_1 + iy_1$ $z = x + iy$

Now $\text{Re}(a+\bar{z}) > \text{Im}(\bar{a}+z)$

$\therefore x_1 + x > -y_1 + y$

$x_1 = 2, y_1 = 10, x = -12, y = 0$

Given inequality is not valid for these values.

S1 is false.

Now $\text{Re}(a+\bar{z}) < \text{Im}(\bar{a}+z)$

$x_1 + x < -y_1 + y$

$x_1 = -2, y_1 = -10, x = 12, y = 0$

Given inequality is not valid for these values.

S2 is false.

19. If $\begin{vmatrix} x+1 & x & x\\ x & x+\lambda & x\\ x & x & x+\lambda^2\end{vmatrix} = \frac{9}{8}(103x+81)$, then $\lambda, \frac{\lambda}{3}$ are the roots of the equation

(1) $4x^2 -24x -27 = 0$ (2) $4x^2 +24x +27 = 0$ (3) $4x^2 -24x +27 = 0$ (4) $4x^2 +24x -27 = 0$

Sol. **(3)**

$$\begin{vmatrix} x+1 & x & x\\ x & x+d & x\\ x & x & x+d^2\end{vmatrix} = \frac{9}{8}(103x+81)$$

Put $\boxed{x=0}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{vmatrix} = \frac{9}{8} \times 81$$

$$\lambda^3 = \frac{9^3}{8}$$

$$\boxed{\lambda = \frac{9}{2}}$$

$$\frac{\lambda}{3} = \frac{9}{2 \times 3} \Rightarrow \frac{3}{2}$$

$$\boxed{\frac{\lambda}{3} = \frac{3}{2}}$$

Option (C) $4x^2 - 24x + 27 = 0$

has Root $\frac{3}{2}, \frac{9}{2}$

20. The domain of the function $f(x) = \frac{1}{\sqrt{[x]^2 - 3[x] - 10}}$ is (where [x] denotes the greatest integer less than or equal to x)

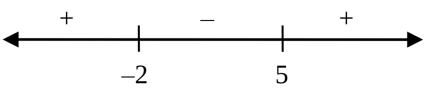
(1) $(-\infty, -3] \cup [6, \infty)$ (2) $(-\infty, -2) \cup (5, \infty)$ (3) $(-\infty, -3] \cup (5, \infty)$ (4) $(-\infty, -2) \cup [6, \infty)$

Sol. **(4)**

$$F(x) = \frac{1}{\sqrt{[x]^2 - 3[x] - 10}}$$

$$[x]^2 - 3[x] - 10 > 0$$

$$([x] + 2)([x] - 5) > 0$$



$[x] < -2$ or $[x] > 5$

$[x] \leq -3$ or $[x] \geq 6$

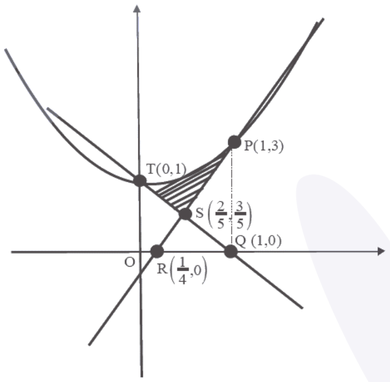
$x < -2$ or $x \geq 6$

$x \in (-\infty, -2) \cup [6, \infty)$

SECTION - B

21. If A is the area in the first quadrant enclosed by the curve C : $2x^2 - y + 1 = 0$, the tangent to C at the point (1,3) and the line x + y = 1, then the value of 60 A is ______.

Sol. 16



$y = 2x^2 + 1$

Tangenet at (1, 3)

$y = 4x - 1$

$A = \int_0^1 (2x^2 + 1)dx - \text{area of } (\Delta QOT) - \text{area of}$

$(\Delta PQR) + \text{area of } (\Delta QRS)$

$A = \left(\frac{2}{3} + 1\right) - \frac{1}{2} - \frac{9}{8} + \frac{9}{40} = \frac{16}{60}$

22. Let A = {1, 2, 3, 4, 5} and B= {1, 2, 3, 4, 5, 6}. Then the number of functions f:A→B satisfying f(1) + f(2) = f(4)–1 is equal to _______.

Sol. 360

$f(1) + f(2) + 1 = f(4) \le 6$

$f(1) + f(2) \le 5$

Case (i) f(1) = 1 ⇒ f(2) = 1, 2, 3, 4 ⇒ 4 mappings

Case (ii) f(1) = 2 ⇒ f(2) = 1, 2, 3 ⇒ 3 mappings

Case (iii) f(1) = 3 ⇒ f(2) = 1, 2 ⇒ 2 mappings

Case (iv) f(1) 4 ⇒ f(2) = 1 ⇒ 1 mapping

f(5) & f(6) both have 6 mappings each

Number of functions = (4 + 3 + 2 + 1) × 6 × 6 = 360

23. Let the tangent to the parabola $y^2 = 12x$ at the point (3, α) be perpendicular to the line 2x+2y = 3. Then the square of distance of the point (6,–4) from the normal to the hyperbola $\alpha^2 x^2 - 9y^2 = 9\alpha^2$ at its point (α –1, α + 2) is equal to _____.

Sol. **116**

$\because P(3,\alpha)$ lies on $y^2 = 12x$

$\Rightarrow \alpha = \pm 6$

But, $\left.\frac{dy}{dx}\right|_{(3,\alpha)} = \frac{6}{\alpha} = 1 \Rightarrow \alpha = 6 \left(\alpha = -6 \text{ reject}\right)$

Now, hyperbola $\frac{x^2}{9} - \frac{y^2}{36} = 1$, normal at

$Q(\alpha - 1, \alpha + 2)$ is $\frac{9x}{5} + \frac{36y}{8} = 45$

$\Rightarrow 2x + 5y - 50 = 0$

Now, distance of (6, –4) from $2x + 5y - 50 = 0$ is equal to

$$\left|\frac{2(6) - 5(4) - 50}{\sqrt{2^2 + 5^2}}\right| = \frac{58}{\sqrt{29}}$$

$\Rightarrow$ Square of distance = 116

24. For $k \in N$, if the sum of the series $1 + \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + \ldots\ldots$ is 10, then the value of k is___

Sol. **2**

$$10 = 1 + \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + \ldots\ldots \text{upto}\infty$$

$$9 = \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + \ldots\ldots \text{upto}\infty$$

$$\frac{9}{k} = \frac{4}{k^2} + \frac{8}{k^3} + \frac{13}{k^4} + \ldots\ldots \text{upto}\infty$$

$$S = 9\left(1 - \frac{1}{k}\right) = \frac{4}{k} + \frac{4}{k^2} + \frac{5}{k^3} + \frac{6}{k^4} \ldots\ldots \text{upto}\infty$$

$$\frac{S}{k} = \frac{4}{k^2} + \frac{4}{k^3} + \frac{5}{k^4} + \ldots\ldots \text{upto}\infty$$

$$\left(1 - \frac{1}{k}\right)S = \frac{4}{k} + \frac{1}{k^3} + \frac{1}{k^4} + \frac{1}{k^5} + \ldots..\infty$$

$$9\left(1 - \frac{1}{k}\right)^2 = \frac{4}{k} + \frac{\frac{1}{k^3}}{\left(1 - \frac{1}{k}\right)}$$

$9(k - 1)^3 = 4k(k - 1) + 1$

$k = 2$

25. Let the line $\ell : x = \frac{1-y}{-2} = \frac{z-3}{\lambda}, \lambda \in R$ meet the plane P : x + 2y + 3z = 4 at the point (α, β, γ). If the angle between the line ℓ and the plane P is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then $\alpha + 2\beta + 6\gamma$ is equal to _____.

Sol. **11**

$\ell : x = \frac{y-1}{2} = \frac{z-3}{\lambda}, \lambda \in \mathbb{R}$

Dr's of line $\ell(1,2,\lambda)$

Dr's of normal vector of plane P : x + 2y + 3z = 4
are (1, 2, 3)
Now, angle between line ℓ and plane P is given by

$$\sin\theta = \left|\frac{1+4+3\lambda}{\sqrt{5+\lambda^2}.\sqrt{14}}\right| = \frac{3}{\sqrt{14}}\left(\text{given } \cos\theta = \sqrt{\frac{5}{14}}\right)$$

$$\Rightarrow \lambda = \frac{2}{3}$$

Let variable point on line ℓ is $\left(t, 2t+1, \frac{2}{3}t+3\right)$

line of plane P.

$\Rightarrow t = -1$

$$\Rightarrow \left(-1,-1,\frac{7}{3}\right) \equiv (\alpha,\beta,\gamma)$$

$$\Rightarrow \alpha + 2\beta + 6\gamma = 11$$

26. The number of points where the curve f(x) = $e^{8x} - e^{6x} - 3e^{4x} - e^{2x} + 1, x \in R$ cuts x-axis, is equal to ________

Sol. **2**

Let $e^{2x} = t$

$\Rightarrow t^4 - t^3 - 3t^2 - t + 1 = 0$

$$\Rightarrow t_2 + \frac{1}{t_2} - \left(t + \frac{1}{t}\right) - 3 = 0$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right) - 5 = 0$$

$$\Rightarrow t + \frac{1}{t} = \frac{1+\sqrt{21}}{2}$$

Two real values of t.

27. If the line l_1 : 3y– 2x = 3 is the angular bisector of the line l_2 : x –y + 1 = 0 and l_3 : ax +βy + 17, then $\alpha^2 + \beta^2 - \alpha - \beta$ is equal to ______.

Sol. **348**

Point of intersection of $\ell_1 : 3y - 2x = 3$

$\ell_2 : x - y + 1 = 0$ is $P \equiv (0,1)$

Which lies on $\ell_3 : \alpha x - \beta y + 17 = 0$,

$\boxed{\Rightarrow \beta = -17}$

Consider a random point $Q \equiv (-1, 0)$

on $\ell_2 : x - y + 1 = 0$, image of Q about

$\ell_2 : x - y + 1 = 0$, is $Q' \equiv \left(\frac{-17}{13}, \frac{6}{13}\right)$ which is calculated by formulae

$$\frac{x-(-1)}{2} = \frac{y-0}{-3} = 2\left(\frac{-2+3}{13}\right)$$

Now, Q' lies in $\ell_3 : \alpha x + \beta y + 17 = 0$

$\boxed{\Rightarrow \alpha = 7}$

Now, $\alpha^2 + \beta^2 - \alpha - \beta = 348$

28. Let the probability of getting head for a biased coin be $\frac{1}{4}$. It is tossed repeatedly until a head appears. Let N be the number of tosses required. If the probability that the equation $64x^2 + 5Nx + 1 = 0$ has no real root is $\frac{p}{q}$, where p and q are co-prime, then q – p is equal to ______.

Sol. **27**

$64x^2 + 5Nx + 1 = 0$

$D = 25N^2 - 256 < 0$

$\Rightarrow N^2 < \frac{256}{25} \Rightarrow N < \frac{16}{5}$

$\therefore N = 1, 2, 3$

$\therefore \text{Probability} = \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{37}{64}$

$\therefore q - p = 27$

29. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c} = 11$, $\vec{b}.(\vec{a} \times \vec{c}) = 27$ and $\vec{b}.\vec{c} = -\sqrt{3}|\vec{b}|$, then $|\vec{a} \times \vec{c}|^2$ is equal to ______.

Sol. **285**

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} + \hat{j} - \hat{k}$

$\vec{b}.(\vec{a} \times \vec{c}) = 27, \vec{a}.\vec{b} = 0$

$\vec{b} \times (\vec{a} \times \vec{c}) = -3\vec{a}$

Let θ be angle between $\vec{b}, \vec{a} \times \vec{c}$

Then $|\vec{b}|.|\vec{a} \times \vec{c}| \sin\theta = 3\sqrt{14}$

$|\vec{b}|.|\vec{a} \times \vec{c}| \cos\theta = 27$

$\Rightarrow \sin\theta = \frac{\sqrt{14}}{\sqrt{95}}$

$\therefore |\vec{b}| \times |\vec{a} \times \vec{c}| = 3\sqrt{95}$

$\Rightarrow |\vec{a} \times \vec{c}| = \sqrt{3} \times \sqrt{95}$

30. Let $S=\left\{z\in C-\{i,2i\}:\frac{z^2+8iz-15}{z^2-3iz-2}\in R\right\}$. If $\alpha-\frac{13}{11}i\in S,\ a\in R-\{0\}$, then $242\alpha^2$ is equal to _______.

Sol. **1680**

$\left(\frac{z^2+8iz-15}{z^2-3iz-2}\right)\in R$

$\Rightarrow 1+\frac{(11iz-13)}{(z^2-3iz-2)}\in R$

Put Z = $\alpha-\frac{13}{11}i$

$\Rightarrow (z^2-3iz-2)$ is imaginary

Put z = x + iy

$\Rightarrow (x^2-y^2+2xyi-3ix+3y-2)\in$ Imaginary

$\Rightarrow \mathrm{Re}\left(x^2-y^2+3y-2+(2xy-3x)i\right)=0$

$\Rightarrow x^2-y^2+3y-2=0$

$x^2=y^2-3y+2$

$x^2=(y-1)(y-2)\therefore z=\alpha-\frac{13}{11}i$

Put $x=\alpha, y=\frac{-13}{11}$

$\alpha^2=\left(\frac{-13}{11}-11\right)\left(\frac{-13}{11}-2\right)$

$\alpha^2=\frac{(24\times 35)}{121}$

$242\alpha^2=48\times 35=1680$