FINAL JEE–MAIN EXAMINATION – APRIL, 2023
Held On Tuesday 11th April, 2023
TIME : 09:00 AM to 12:00 PM

SECTION-A

1. Let $x_1, x_2, \ldots, x_{100}$ be in an arithmetic progression, with $x_1 = 2$ and their mean equal to 200. If $y_i = i(x_i - i)$, $1 \le i \le 100$, then the mean of $y_1, y_2, \ldots, y_{100}$ is :

(1) 10051.50 (2) 10100 (3) 10101.50 (4) 10049.50

Sol. **(4)**

Mean = 200

$$\Rightarrow \frac{\frac{100}{2}(2\times2+99d)}{100} = 200$$

$$\Rightarrow 4 + 99d = 400$$

$$\Rightarrow d = 4$$

$$y_i = i(xi - 1)$$

$$= i(2+(i-1)4-i) = 3i^2 - 2i$$

$$\text{Mean } = \frac{\sum y_i}{100}$$

$$= \frac{1}{100}\sum_{i=1}^{100} 3i^2 - 2i$$

$$= \frac{1}{100}\left\{\frac{3\times100\times101\times201}{6} - \frac{2\times100\times101}{2}\right\}$$

$$= 101\left\{\frac{201}{2} - 1\right\} = 101 \times 99.5$$

= 10049.50

2. The number of elements in the set $S = \{\theta \in [0, 2\pi] : 3\cos^4\theta - 5\cos^2\theta - 2\sin^6\theta + 2 = 0\}$ is :

(1) 10 (2) 9 (3) 8 (4) 12

Sol. **(2)**

$$3\cos^4\theta - 5\cos^2\theta - 2\sin^6\theta + 2 = 0$$

$$\Rightarrow 3\cos^4\theta - 3\cos^2\theta - 2\cos^2\theta - 2\sin^6\theta + 2 = 0$$

$$\Rightarrow 3\cos^4\theta - 3\cos^2\theta + 2\sin^2\theta - 2\sin^6\theta = 0$$

$$\Rightarrow 3\cos^2\theta(\cos^2\theta - 1) + 2\sin^2\theta(\sin^4\theta - 1) = 0$$

$$\Rightarrow -3\cos^2\theta\sin^2\theta + 2\sin^2\theta(1+\sin^2\theta)\cos^2\theta - 1$$

$$\Rightarrow \sin^2\theta\cos^2\theta(2 + 2\sin^2\theta - 3) = 0$$

$$\Rightarrow \sin^2\theta\cos^2\theta(2\sin^2\theta - 1) = 0$$

$$(C1)\sin^2\theta = 0 \rightarrow 3\,\text{solution};\ \theta = \{0, \pi, 2\pi\}$$

$$(C2)\cos^2\theta = 0 \rightarrow 2\ \text{solution};\ \theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

$$(C3)\ \sin^2\theta = \frac{1}{2} \rightarrow 4\ \text{solution};\ \theta = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$

No. of solution = 9

3. The value of the integral $\int_{-\log_e 2}^{\log_e 2} e^x\left(\log_e\left(e^x+\sqrt{1+e^{2x}}\right)\right)dx$ is equal to :

(1) $\log_e\left(\frac{(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right)+\frac{\sqrt{5}}{2}$

(2) $\log_e\left(\frac{2(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right)-\frac{\sqrt{5}}{2}$

(3) $\log_e\left(\frac{\sqrt{2}(3-\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right)+\frac{\sqrt{5}}{2}$

(4) $\log_e\left(\frac{\sqrt{2}(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}}\right)-\frac{\sqrt{5}}{2}$

Sol. **(4)**

$$I=\int_{-\ln 2}^{\ln 2} e^x\left(\ln\left(e^x+\sqrt{1+e^{2x}}\right)\right)dx$$

Put $e^x=t\Rightarrow e^x dx=dt$

$$I=\int_{1/2}^{2}\ln\left(t+\sqrt{1+t^2}\right)dt$$

Applying integration by parts.

$$=\left[t\ln\left(t+\sqrt{1+t^2}\right)\right]_{\frac{1}{2}}^{2}-\int_{1/2}^{2}\frac{t}{t+\sqrt{1+t^2}}\left(1+\frac{2t}{2\sqrt{1+t^2}}\right)dt$$

$$=2\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\int_{1/2}^{2}\frac{t}{\sqrt{1+t^2}}dt$$

$$=2\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\frac{\sqrt{5}}{2}$$

$$=\ln\left(\frac{(2+\sqrt{5})^2}{\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}}\right)-\frac{\sqrt{5}}{2}$$

4. Let S = {M = [a_{ij}], $a_{ij}\in\{0, 1, 2\}$, $1\le i, j\le 2$} be a sample space and A = {M $\in$ S : M is invertible} be an event. Then P(A) is equal to :

(1) $\frac{16}{27}$ (2) $\frac{50}{81}$ (3) $\frac{47}{81}$ (4) $\frac{49}{81}$

Sol. **(2)**

M $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$, where a, b, c, d, $\in\{0,1,2\}$

n(s) = 3^4 = 81

we first bound p $(\bar{A})$

$|m|=0\Rightarrow ad=bc$

$ad=bc=0\Rightarrow$ no. of (a, b, c, d) = $(3^2-2^2)^2=25$

ad = bc =1 $\Rightarrow$ no. of (a,b,c,d) = 1^2 = 1

ad = bc = 2 $\Rightarrow$ no. of (a,b,c,d) = 2^2 = 4

ad = bc = 4 $\Rightarrow$ no. of (a,b,c,d) = 1^2 = 1

$:P(\bar{A})=\frac{31}{81}\Rightarrow p(A)=\frac{50}{81}$

5. Let $f : [2, 4] \to \mathbb{R}$ be a differentiable function such that $(x \log_e x) f'(x) + (\log_e x) f(x) + f(x) \geq 1$, $x \in [2, 4]$ with $f(2) = \frac{1}{2}$ and $f(4) = \frac{1}{4}$. Consider the following two statements :

(A) : $f(x) \leq 1$, for all $x \in [2, 4]$

(B) : $f(x) \geq \frac{1}{8}$, for all $x \in [2, 4]$

Then,

(1) Only statement (B) is true

(2) Only statement (A) is true

(3) Neither statement (A) nor statement (B) is true

(4) Both the statements (A) and (B) are true

Sol. **(4)**

$x\ell nxf'(x) + \ln xf(x) + f(x) \geq 1, x \in [2,4]$

And $f(2) = \frac{1}{2}$, $f(4) = \frac{1}{4}$

Now xlnx, $\frac{dy}{dx} + (\ln + 1)y \geq 1$

$\frac{d}{dx}(y \cdot x\ln x) \geq 1$

$\frac{d}{dx}(f(x).x\ln x) \geq 1$

$\Rightarrow \frac{d}{dx}(x\ln xf(x) - x) \geq 0, x \in [2,4]$

$\Rightarrow$ The function $g(x) = x \ln x f(x) - x$ is increasing in [2,4]

And $g(2) = 2 \ln 2f(2) - 2 = \ln 2 - 2$

$g(2) = 4 \ln 4f(4) - 4 = \ln 4 - 4$

$= 2(\ln 2 - 2)$

Now $g(2) \leq g(x) \leq g(4)$

$\ln 2 - 2 \leq x \ln x f(x) - x \leq 2(\ln 2 - 2)$

$\frac{\ln 2 - 2}{x\ln x} + \frac{1}{\ln x} \leq f(x) \leq \frac{2(\ln 2 - 2)}{x \ln x} + \frac{1}{\ln x}$

Now for $x \in [2,4]$

$\frac{2(\ell n2 - 2)}{x\ln x} + \frac{1}{\ell nx} < \frac{2(\ln 2 - 2)}{2\ln 2} + \frac{1}{\ln 2} = 1 - \frac{1}{\ln 2} < 1$

$\Rightarrow f(x) \leq 1$ for $x \in [2,4]$

$\frac{\ln 2 - 2}{x\ln x} + \frac{1}{\ln x} \geq \frac{\ln 2 - 2}{4\ln 4} + \frac{1}{\ln 4} = \frac{1}{8} + \frac{1}{2\ln 2} > \frac{1}{8}$

$\Rightarrow f(x) \geq \frac{1}{8}$ for $x \in [2,4]$

Hence both A and B are true.

6. Let A be a 2 × 2 matrix with real entries such that A' = α A + I, where a ∈ ℝ – {–1, 1}. If det (A^2 – A) = 4, then the sum of all possible values of α is equal to :

(1) 0 (2) $\frac{5}{2}$ (3) 2 (4) $\frac{3}{2}$

Sol. **(2)**

$A^T = \alpha A + I$

$A = \alpha A^T + I$

$A = \alpha(\alpha A + I) + I$

$A = \alpha^2 A + (\alpha+1)I$

$A(1-\alpha^2) = (\alpha+1)I$

$A = \frac{I}{1-\alpha}$ … (1)

$|A| = \frac{1}{(1-\alpha)^2}$ … (2)

$|A^2 - A| = |A||A-1|$ … (3)

$A - I = \frac{I}{I-\alpha} - I = \frac{\alpha}{1-\alpha}I$

$|A - I| = \left(\frac{\alpha}{1-\alpha}\right)^2$ … (4)

Now $|A^2 - A| = 4$

$|A||A-I| = 4$

$\Rightarrow \frac{1}{(1-\alpha)^2}\frac{\alpha^2}{(1-\alpha^2)} = 4$

$\Rightarrow \frac{\alpha}{(1-\alpha)^2} = \pm 2$

$\Rightarrow 2(1-\alpha)^2 = \pm\alpha$

$(C_1)\, 2(1-\alpha)^2 = \alpha$ $(C_2)\, 2(1-\alpha)^3 = -\alpha$

$2\alpha^2 - 5\alpha + 2 = 0 < \alpha_1, \alpha_2$ $2\alpha^2 - 3\alpha + 2 = 0$

$\alpha_1 + \alpha_2 = \frac{5}{2}$ $\alpha \notin R$

7. The number of integral solutions x of $\log_{\left(x+\frac{7}{2}\right)}\left(\frac{x-7}{2x-3}\right)^2 \geq 0$ is :

(1) 5 (2) 7 (3) 8 (4) 6

Sol. **(4)**

$\log_{x+\frac{7}{2}}\left(\frac{x-7}{2x-3}\right)^2 \geq 0$

Feasible region: $x + \frac{7}{2} > 0 \Rightarrow x > -\frac{7}{2}$

And $x + \frac{7}{2} \neq 1 \Rightarrow x \neq \frac{-5}{2}$

And $\frac{x-7}{2x-3} \neq 0$ and $2x-3 \neq 0$

$\Downarrow$ $\Downarrow$

$x \neq 7$ $x \neq \frac{3}{2}$

Taking intersection: $x \in \left(\frac{-7}{2}, \infty\right) - \left\{-\frac{5}{2}, \frac{3}{2}, 7\right\}$

Now $\log_a b \geq 0$ if $a > 1$ and $b \geq 1$

$a \in (0,1)$ and $b \in (0,1)$

$C - I;\ x + \frac{7}{2} > 1$ and $\left(\frac{x-7}{2x-3}\right)^2 \geq 1$

$x > -\frac{5}{2}$;$(2x-3)^2 - (x-7)^2 \leq 0$

$(2x-3+x-7)\ (2x-3-x+7) \leq 0$

$(3x-10)(x+4) \leq 0$

$x \in \left[-4, \frac{10}{3}\right]$

Intersection: $x \in \left(\frac{-5}{2}, \frac{10}{3}\right]$

$C - II:\ x + \frac{7}{2} \in (0,1)$ and $\left(\frac{x-7}{2x-3}\right)^2 \in (0,1)$

$0 < x + \frac{7}{2} < 1;\ \left(\frac{x-7}{2x-3}\right)^2 < 1$

$-\frac{7}{2} < x < \frac{-5}{2}$; $(x-7)^2 < (2x-3)^2$

$x \in (-\infty, -4) \cup \left(\frac{10}{3}, \infty\right)$

No common values of x.

Hence intersection with feasible region

We get $x \in \left(\frac{-5}{2}, \frac{10}{3}\right] - \left\{\frac{3}{2}\right\}$

Integral value of x are {–2, –1, 0, 1, 2, 3}

No. of integral values = 6

8. For any vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, with $10\,|a_i| < 1$, i = 1, 2, 3, consider the following statements :

(A) : $\max\{|a_1|, |a_2|, |a_3|\} \leq |\vec{a}|$

(B) : $|\vec{a}| \leq 3\max\{|a_1|, |a_2|, |a_3|\}$

(1) Only (B) is true

(2) Both (A) and (B) are true

(3) Neither (A) nor (B) is true

(4) Only (A) is true

Sol. **(2)**

Without loss of generality

Let $|a_1| \leq |a_2| \leq |a_3|$

$|\vec{a}|^2 = |\vec{a}_1|^2 + |\vec{a}_2|^2 + |\vec{a}_3|^2 \geq (a_3)^2$

$\Rightarrow |\vec{a}| \geq |a_3| = \max\{|a_1|, |a_2|, |a_3|\}$

A is true

$|\vec{a}|^2 = |a_1|^2 + |a_2|^2 + |a_3|^2 \leq |a_3|^2 + |a_3|^2 + |a_3|^2$

$\Rightarrow |\vec{a}|^2 \leq 3|a_3|^2$

$\Rightarrow |\vec{a}| \leq \sqrt{3}|\vec{a}_3| = \sqrt{3}\ \max\{|a_1|, |a_2|, |a_3|\}$

$\leq 3\max\{|a_1|, |a_2|, |a_3|\}$

(2) is true

9. The number of triplets (x,y,z), where x, y, z are distinct non negative integers satisfying x + y + z = 15, is :

(1) 136 (2) 114 (3) 80 (4) 92

Sol. **(2)**

x + y + z = 15

Total no. solution = ${}^{15+3-1}C_3 = 136 \ldots (1)$

Let x = y ≠ z

2x+ z = 15 ⇒ z = 15 – 2t

$\Rightarrow r \in \{0, 1, 2, \ldots 7\} - \{5\}$

∴ 7 solutions

∴ there are 21 solutions in which exactly

Two of x, y, z are equal … (2)

There is one solution in which x = y = z … (3)

Required answer = 136 – 21 – 1= 144

10. Let sets A and B have 5 elements each. Let mean of the elements in sets A and B be 5 and 8 respectively and the variance of the elements in sets A and B be 12 and 20 respectively. A new set C of 10 elements is formed by subtracting 3 from each element of A and adding 2 to each element of B. Then the sum of the mean and variance of the elements of C is _______.

(1) 36 (2) 40 (3) 32 (4) 38

Sol. **(4)**

$\omega\ A = \{a_1, a_2, a_3, a_4, a_5\}$

$B = \{b_1, b_2, b_3, b_4, b_5\}$

Given, $\sum_{i=1}^{5} ai = 25,\ \sum_{i=1}^{5} bi = 40$

$$\frac{\sum_{i=1}^{5} a_i^2}{5} - \left(\frac{\sum_{i=1}^{5} a_i}{5}\right)^2 = 12,\ \frac{\sum_{i=1}^{5} b_i^2}{5} - \left(\frac{\sum_{i=1}^{5} b_i}{5}\right)^2 = 20$$

$$\Rightarrow \sum_{i=1}^{5} a_i^2 = 185,\quad \sum_{i=1}^{5} b_i^2 = 420$$

Now, $C = \{C_1, C_2, \ldots C_{10}\}$

$$\therefore \text{ Mean of C, } \bar{C} = \frac{\left(\sum a_i - 15\right) + \left(\sum b_i - 10\right)}{10}$$

$$\bar{C} = \frac{10 + 50}{10} = 6$$

$$\therefore \sigma^2 = \frac{\sum_{i=1}^{10} C_i^2}{10} = (\bar{C})^2$$

$$= \frac{\sum(a_i - 3)^2 + \sum(b_i - 2)^2 +}{10} - (6)^2$$

$$= \frac{\sum a_i^2 + \sum b_i^2 - 6\sum a_i + 4\sum b_i + 65}{10} - 36$$

$$= \frac{185 + 420 - 150 + 160 + 65}{10} - 36$$

= 32

$\therefore$ Mean + Variance = $\bar{C} + \sigma^2 = 6 + 32 = 38$

11. Area of the region $\{(x, y) : x^2 + (y - 2)^2 \le 4, x^2 \ge 2y\}$ is :

(1) $\pi + \frac{8}{3}$ (2) $2\pi + \frac{16}{3}$ (3) $2\pi - \frac{16}{3}$ (4) $\pi - \frac{8}{3}$

Sol. **(3)**

$x^2 + (y-2)^2 \le 2^2$ and $x^2 \ge 2y$

Solving circle and parabola simultaneously:

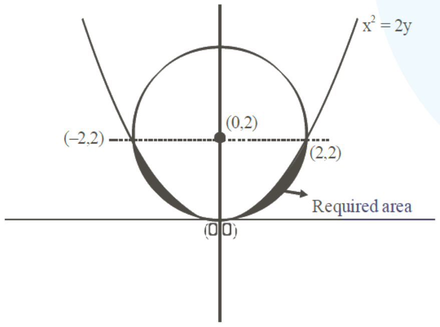
$2y + y^2 - 4y + 4 = 4$

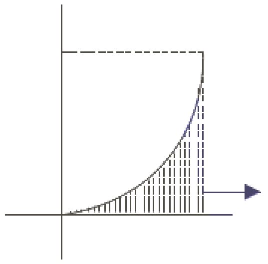
$y^2 - 2y = 0$

y = 0, 2

Put $y = 2$ in $x^2 = 2y \rightarrow x = \pm 2$

$\Rightarrow$ (2, 2) and (–2, 2)





$$= 2 \times 2 - \frac{1}{4} \cdot \pi \cdot 2^2 = 4 - \pi$$

Required area = $2\left[\int_0^2 \frac{x^2}{2} dx - (4 - \pi)\right]$

$$= 2\left[\frac{x^3}{6}\Big|_0^2 - 4 + \pi\right]$$

$$= 2\left[\frac{4}{3} + \pi - 4\right]$$

$$= 2\left[\pi - \frac{8}{3}\right]$$

$$= 2\pi - \frac{16}{6}$$

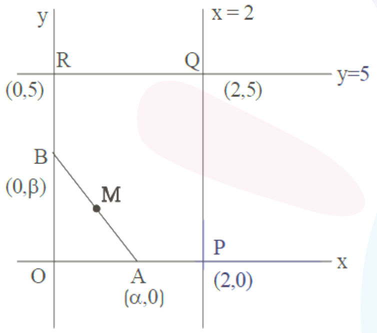
12. Let R be a rectangle given by the line x = 0, x = 2, y = 0 and y = 5. Let A (α, 0) and B (0, β), α ∈ [0, 2] and β ∈ [0, 5], be such that the line segment AB divides the area of the rectangle R in the ratio 4 : l. Then, the mid-point of AB lies on a :

(1) straight line (2) parabola (3) circle (4) hyperbola

Sol. **(4)**

$$\frac{\text{ar}(\text{OPQR})}{\text{or}(\text{OAB})} = \frac{4}{1}$$

Let M be the mid-point of AB.



$$M(h,k) \equiv \left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

$$\Rightarrow \frac{10 - \frac{1}{2}\alpha\beta}{\frac{1}{2}\alpha\beta} = 4$$

$$\Rightarrow \frac{5}{2}\alpha\beta = 10 \Rightarrow \alpha\beta = 4$$

$$\Rightarrow (2h)(2K) = 4$$

∴ Locus of M is xy = 1

Which is a hyperbola.

13. Let $\vec{a}$ be a non-zero vector parallel to the line of intersection of the two places described by $\hat{i} + \hat{j}, \hat{i} + \hat{k}$ and $\hat{i} - \hat{j}, \hat{j} - \hat{k}$. If θ is the angle between the vector $\vec{a}$ and the vector $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{a}.\vec{b} = 6$, then ordered pair $\left(\theta, |\vec{a} \times \vec{b}|\right)$ is equal to :

(1) $\left(\frac{\pi}{3}, 6\right)$ (2) $\left(\frac{\pi}{4}, 3\sqrt{6}\right)$ (3) $\left(\frac{\pi}{3}, 3\sqrt{6}\right)$ (4) $\left(\frac{\pi}{4}, 6\right)$

Sol. **(4)**

$\vec{n}_1$ and $\vec{n}_2$ are normal vector to the plane $\hat{i} + \hat{j}, \hat{i} + \hat{k}$ and $\hat{i} - \hat{j}, \hat{i} - \hat{k}$ respectively

$$\vec{n}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j} - \hat{k}$$

$$\vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{a} = \lambda |\vec{n}_2 \times \vec{n}_2|$$

$$= \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \lambda(-2\hat{j} + 2\hat{k})$$

$$\vec{a} \cdot \vec{b} = \lambda |0 + 4 + 2| = 6$$

$$\Rightarrow \lambda = 1$$

$$\vec{\alpha} = -2\hat{j} + 2\hat{k}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|a||b|}$$

$$\cos\theta = \frac{6}{2\sqrt{2} \times 3} = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

Now $|\vec{a}.\vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = |a|^2 |b|^2$

$$36 + |\vec{a} \times b^2| = 8 \times 9 = 72$$

$$|\vec{a} \times b|^2 = 36$$

$$|\vec{a} \times \vec{b}| = 6$$

14. Let w_1 be the point obtained by the rotation of $z_1 = 5 + 4i$ about the origin through a right angle in the anticlockwise direction, and w_2 be the point obtained by the rotation of $z_2 = 3 + 5i$ about the origin through a right angle in the clockwise direction. Then the principal argument of $w_1 - w_2$ is equal to :

(1) $\pi - \tan^{-1}\frac{8}{9}$ (2) $-\pi + \tan^{-1}\frac{8}{9}$ (3) $\pi - \tan^{-1}\frac{33}{5}$ (4) $-\pi + \tan^{-1}\frac{33}{5}$

Sol. **(1)**

$$W_1 = z_1 i = (5 + 4i)i = -4 + 5i \ \ldots \text{(i)}$$

$$W_1 = z_2(-i) = (3 + 5i)(-i) = 5 - 3i \ \ldots (2)$$

$$W_1 - W_2 = -9 + 8i$$

Principal argument = $\pi - \tan^{-1}\left(\frac{8}{9}\right)$

15. Consider ellipse $E_k : kx^2 + k^2y^2 = 1, k = 1, 2, \ldots, 20$. Let C_k be the circle which touches the four chords joining the end points (one on minor axis and another on major axis) of the ellipse E_k. If r_k is the radius of the circle C_k then the value of $\sum_{k=1}^{20} \frac{1}{{r_k}^2}$ is

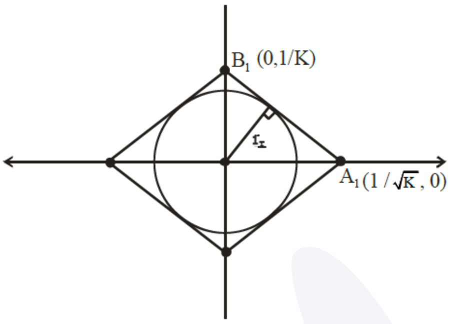
(1) 3320 (2) 3210 (3) 3080 (4) 2870

Sol. **(3)**

$$Kx^2 + K^2y^2 = 1$$

$$\frac{x^2}{1/K}+\frac{y^2}{1/K^2}=1$$

Now



Equation of

$$A_1B_2;\frac{x}{1/\sqrt{K}}+\frac{y}{1/K}=1\Rightarrow\sqrt{K}x+Ky=1$$

$r_K=\perp r$ distance of (0, 0) from line A_1B_1

$$r_K=\left|\frac{(0+0-1)}{\sqrt{K+K^2}}\right|=\frac{1}{\sqrt{K+K^2}}$$

$$\frac{1}{r_K^2}=K+K^2\Rightarrow\sum_{k=1}^{20}\frac{1}{r_K^2}=\sum_{K=1}^{20}(K+K^2)$$

$$=\sum_{K=1}^{20}K+\sum_{K=1}^{20}K^2$$

$$=\frac{20\times 21}{2}+\frac{20.21.41}{6}$$

= 210 + 10 × 7 × 41

= 210 + 2870

= 3080

16. If equation of the plane that contains the point (–2,3,5) and is perpendicular to each of the planes 2x + 4y + 5z = 8 and 3x – 2y + 3z = 5 is $\alpha x + \beta y + \gamma z + 97 = 0$ then $\alpha + \beta + \gamma$ = :

(1) 15 (2) 18 (3) 17 (4) 16

Sol. **(1)**

The equation of plane through (–2,3,5) is

a(x+2) + b(y – 3) + c (z – 5) = 0

it is perpendicular to 2x + 4y + 5z = 8 & 3x – 2y + 3z = 5

$\therefore$ 2a + 4b + 5c =0

3a – 2b + 3c = 0

$$\therefore\frac{a}{\begin{vmatrix}4 & 5\\ -2 & 3\end{vmatrix}}=\frac{-b}{\begin{vmatrix}2 & 5\\ 3 & 3\end{vmatrix}}=\frac{c}{\begin{vmatrix}2 & 4\\ 3 & -2\end{vmatrix}}$$

$$\Rightarrow\frac{a}{22}=\frac{b}{9}=\frac{c}{-16}$$

$\therefore$ Equation of plane is

22(x + 2) + 9(y – 3) –16(z – 5) = 0

$\Rightarrow$ 22x + 9y – 16z + 97 = 0

Comparing with $\alpha x+\beta y+\gamma x+97=0$

We get $\alpha+\beta+\gamma=22+9-16=15$

17. An organization awarded 48 medals in event 'A', 25 in event 'B' and 18 in event 'C'. If these medals went to total 60 men and only five men got medals in all the three events, then, how many received medals in exactly two of three events ?

(1) 15 (2) 9 (3) 21 (4) 10

Sol. **(3)**

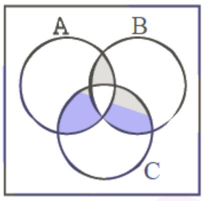
$|A| = 48$

$|B| = 25$

$|C| = 18$

$|A \cup B \cup C| = 60$ [Total]

$|A \cap B \cap C| = 5$



$|A \cap B \cap C| = \sum |A| - \sum |A \cap B| + |A \cap B \cap C|$

$\Rightarrow \sum |A \cap B| = 48 + 25 + 18 + 5 - 60$

$= 36$

No. of men who received exactly 2 medals

$\Rightarrow \sum |A \cap B| - 3|A \cap B \cap C|$

$= 36 - 15$

$= 21$

18. Let y = y(x) be a solution curve of the differential equation. $(1 - x^2y^2)dx = ydx + xdy$. If the line x = 1 intersects the curve y = y(x) at y = 2 and the line x = 2 intersects the curve y = y(x) at y = α, then a value of α is :

(1) $\frac{1+3e^2}{2(3e^2-1)}$ (2) $\frac{1-3e^2}{2(3e^2+1)}$ (3) $\frac{3e^2}{2(3e^2-1)}$ (4) $\frac{3e^2}{2(3e^2+1)}$

Sol. **(1)**

$(1 - x^2y^2)dx = ydx + xdy, y(1) = 2$

$y(2) = \infty$

$dx = \frac{d(xy)}{1-(xy)^2}$

$\int dx = \int \frac{d(xy)}{1-(xy)^2}$

$x = \frac{1}{2}\ln\left|\frac{1+xy}{1-xy}\right| + C$

Put x = 1 and y = 2:

$1 = \frac{1}{2}\ln\left|\frac{1+2}{1-2}\right| + C$

$C = 1 - \frac{1}{2}\ln 3$

Now put x = 2 :

$$2=\frac{1}{2}\ln\left|\frac{1+2\alpha}{1-2\alpha}\right|+1-\frac{1}{2}\ln 3$$

$$1+\frac{1}{2}\ln 3=\frac{1}{2}\left|\frac{1+2\alpha}{1-2\alpha}\right|$$

$$2+\ln 3=\left|\frac{1+2\alpha}{1-2\alpha}\right|$$

$$\left|\frac{1+2\alpha}{1-2\alpha}\right|=3e^2$$

$$\frac{1+2\alpha}{1-2\alpha}=3e^2,-3e^2$$

$$\frac{1+2\alpha}{1-2\alpha}=3e^2\Rightarrow\alpha=\frac{3e^2-1}{2\left(3e^2+1\right)}$$

And $\frac{1+2\alpha}{1-2\alpha}=-3e^2\Rightarrow\alpha=\frac{3e^2+1}{2\left(3e^2-1\right)}$

19. Let (α, β, γ) be the image of the point P (2, 3, 5) in the plane 2x + y – 3z = 6. Then α + β + γ is equal to :

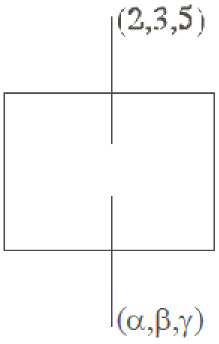
(1) 5 (2) 9 (3) 10 (4) 12

Sol. **(3)**

$$\frac{\alpha-2}{2}=\frac{\beta-3}{1}=\frac{\gamma-5}{-3}=-2\left(\frac{2x2+3-3\times5-6}{2^2+1^2+1-3^2}\right)=2$$

$\frac{\alpha-2}{2}=2$ $\beta-3=2$ $\gamma-5=-6$

$\alpha=6$ $\beta=5$ $\gamma=-1$



$\alpha+\beta+\gamma=10$

20. Let f(x) = [x^2 – x] + |–x+[x]|, where x ∈ ℝ and [t] denotes the greatest integer less than or equal to t. Then, f is :

(1) not continuous at x = 0 and x = 1

(2) continuous at x = 0 and x = 1

(3) continuous at x = 1, but not continuous at x = 0

(4) continuous at x = 0, but not continuous at x = 1

Sol. **(3)**

Here f(x) = [x(x – 1)] + {x}

$f\left(0^+\right)=-1+0=-1$ $f\left(1^+\right)=0+0=0$

$f\left(0\right)=0$ $f\left(1\right)=0$

$f\left(1^-\right)=-1+1=0$

$\therefore$ f(x) is continuous at x = 1, discontinuous at x = 0

SECTION-B

21. The number of integral terms in the expansion of $\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is equal to :

Sol. **(171)**

The number of integral term in the expression of

$\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is equal to

General term $= {}^{680}C_r\left(3^{\frac{1}{2}}\right)^{680-r}\left(5^{\frac{1}{4}}\right)^{r}$

$= {}^{680}C_r 3^{\frac{680-r}{2}} 5^{\frac{r}{4}}$

Values' s of r, where $\frac{r}{4}$ goes to integer

r = 0, 4, 8, 12, …… 680

All value of r are accepted for $\frac{680-r}{2}$ as well so

No of integral terms = 171.

22. The number of ordered triplets of the truth values of p, q and r such that the truth value of the statement $(p \vee q) \wedge (p \vee r) \Rightarrow (q \vee r)$ is True, is equal to ______:

Sol. **(7)**

p	**q**	**r**	**Pvq**	**Pvr**	**(pvq) ∧ (pvr)**	**qvr**	**(pvq) ∧ (pvr) → qvr**
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	F	T	F	T	T
F	F	F	F	F	F	F	T

Hence total no of ordered triplets are 7

23. Let $A=\begin{bmatrix}0 & 1 & 2\\ a & 0 & 3\\ 1 & c & 0\end{bmatrix}$, where a, c $\in \mathbb{R}$. If A^3 = A and the positive value of a belongs to the interval (n – 1, n], where n $\in \mathbb{N}$, then n is equal to ______:

Sol. **(2)**

$$A=\begin{bmatrix}0&1&2\\a&0&3\\1&c&0\end{bmatrix}$$

$A^3 = A$

$$A^2=\begin{bmatrix}0&1&2\\a&0&3\\1&c&0\end{bmatrix}\begin{bmatrix}0&1&2\\a&0&3\\1&c&0\end{bmatrix}$$

$$A^2=\begin{bmatrix}a+2&2c&3\\3&a+3c&2a\\ac&1&2+3c\end{bmatrix}$$

$$A^3=\begin{bmatrix}a+2&2c&3\\3&a+3c&2a\\ac&a&2+3c\end{bmatrix}\begin{bmatrix}0&1&2\\a&0&3\\1&c&0\end{bmatrix}$$

$$A^3=\begin{bmatrix}2ac+3&a+2+3c&2a+4+6c\\a(a+3c)+2a&3+2ac&6+3a+9c\\a+2+3c&ac+c(2+3c)&2ac+3\end{bmatrix}$$

Given $A^3 = A$

$2ac + 3 = 0$ … (1) and $a + 2 + 3c = 1$

$a + 1 + 3c = 0$

$a + 1 - \dfrac{9}{2a} = 0$

$2a^2 + 2a - 9 = 0$

$f(1) < 0,\ f(2) > 0$

$a \in (1,2]$

$n = 2$

24. For m, n > 0, let $\alpha(m, n) = \int_0^2 t^m(1+3t)^n\,dt$. If $11\alpha(10, 6) + 18\alpha(11, 5) = p\,(14)^6$, then p is equal to ______:

Sol. **(32)**

$$\alpha(m,n)=\int_0^2 t^m(1+3t)^n\,dt$$

If $11\alpha(10,6)+18\alpha(11,5)=p(14)^6$ then P

$$=11\int_0^2 \underset{II}{t^{10}}\,\underset{I}{(1+3t)^6}+10\int^2 t^{11}(1+3t)^5\,dt$$

$$=11\left[(1+3t)^6\cdot\frac{t^{11}}{11}-\int 6(1+3t)^5\cdot 3\frac{t^{11}}{11}\right]_0^2+18\int_0^2 t^{11}(1+3t)^5\,dt$$

$$=\left(t^{11}(1+3t)^6\right)_0^2$$

$$=2^{11}(7)^6$$

$$=2^5(14)^6$$

$$=32(14)^6$$

25. Let S = $S=109+\frac{108}{5}+\frac{107}{5^2}+....+\frac{2}{5^{107}}+\frac{1}{5^{108}}$. Then the value of $(16S-(25)^{-54})$ is equal to _____:

Sol. **(2175)**

$$S=109+\frac{108}{5}+\frac{107}{5^2}......+\frac{1}{5^{108}}$$

$$\frac{S}{5}=\frac{109}{5}+\frac{108}{5^2}......\frac{2}{5^{108}}+\frac{1}{5^{109}}$$

$$\frac{4S}{5}=109-\frac{1}{5}-\frac{1}{5^2}......-\frac{1}{5^{108}}-\frac{1}{5^{109}}$$

$$=109-\left(\frac{\frac{1}{5}\left(1-\frac{1}{5^{109}}\right)}{\left(1-\frac{1}{5}\right)}\right)$$

$$=109-\frac{1}{4}\left(1-\frac{1}{5^{109}}\right)$$

$$=109-\frac{1}{4}+\frac{1}{4}\times\frac{1}{5^{109}}$$

$$s=\frac{5}{4}\left(109-\frac{1}{4}+\frac{1}{4.5^{109}}\right)$$

$$16S = 20\times109-5+\frac{1}{5^{108}}$$

$$16S-(25)^{-54} = 2180-5 = 2175$$

26. Let H $H_n:\frac{x^2}{1+n}-\frac{y^2}{3+n}=1, n\in N$. Let k be the smallest even value of n such that the eccentricity of H_k is a rational number. If l is the length of the latus rectum of H_k, then 21 l is equal to ____:

Sol. **(306)**

$$Hn\Rightarrow\frac{x^2}{1+n}-\frac{y^2}{3+n}=1$$

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{3+n}{1+n}}=\sqrt{\frac{2n+4}{n+1}}$$

$$e=\sqrt{\frac{2n+4}{n+1}}$$

n = 48 (smallest even value for which $e\in Q$)

$$e=\frac{10}{7}$$

$a^2=n+1$ $\quad$ $b^2=n+3$

$=49$ $\quad$ $=51$

l =length of LR = $\frac{2b^2}{a}$

$$L = 2\cdot\frac{51}{7}$$

$$l=\frac{102}{7}$$

$$21\ell=306$$

27. The mean of the coefficients of x, x^2, …. x^7 in the binomial expansion of $(2 + x)^9$ is _____:

Sol. **2736**

Coefficient of x = ${}^9C_1 2^8$

Coef. $x^2 = {}^9C_2 2^7$

Coef. $x^7 = {}^9C_7 \cdot 2^2$

$$\text{Mean} = \frac{{}^9C_1 \cdot 2^8 + {}^9C_2 \cdot 2^7 \ldots + {}^9C_7 \cdot 2^2}{7}$$

$$= \frac{(1+2)^9 - {}^9C_0 \cdot 2^9 - {}^9C_8 \cdot 2^1 - {}^9C_9}{7}$$

$$= \frac{3^9 - 2^9 - 18 - 1}{7}$$

$$= \frac{19152}{7} = 2736$$

28. If a and b are the roots of the equation $x^2 - 7x - 1 = 0$, then the value of $\frac{a^{21} + b^{21} + a^{17} + b^{17}}{a^{19} + b^{19}}$ is equal to _____:

Sol. **(51)**

$x^2 - 7x - 1 = 0 <$ a, b

By newton's theorem

$$S_{n+2} - 7S_{n+1} - S_n = 0$$

$$S_{21} - 7S_{20} - S_{19} = 0$$

$$S_{20} - 7S_{19} - S_{18} = 0$$

$$S_{19} - 7S_{18} - S_{17} = 0$$

$$\frac{S_{21} + S_{17}}{S_{19}} = \frac{S_{21} + (S_{19} - 7S_{18})}{S_{19}}$$

$$= \frac{50S_{19} + (S_{21} - 7S_{20})}{S_{19}}$$

$$= 51 \cdot \frac{S_{19}}{S_{19}} = 51$$

29. In an examination, 5 students have been allotted their seats as per their roll numbers. The number of ways, in which none of the students sits on the allotted seat, is _______.

Sol. **(44)**

Derangement of 5 students

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$

= 60 – 20 + 5 – 1

= 40+ 4

= 44

30. Let a line l pass through the origin and be perpendicular to the lines

$l_1 : \vec{r} = (\hat{i} - 11\hat{j} - 7\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}), \lambda \in \mathbb{R}$ and

$l_2 : \vec{r} = (-\hat{i} + \hat{k}) + \mu(2\hat{i} + 2\hat{j} + \hat{k}), \mu \in \mathbb{R}.$

If P is the point of intersection of l and l_1, and Q (α, β, γ) is the foot of perpendicular from P on l_2 , then $9(\alpha + \beta + \gamma)$ is equal to _____:

Sol. **(5)**

Let $\ell = (0\hat{i} + 0\hat{j} + 0\hat{k}) + \gamma(a\hat{i} + b\hat{j} + c\hat{k})$

$= \gamma(a\hat{i} + b\hat{j} + c\hat{k})$

$$a\hat{i} + b\hat{j} + c\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 2 & 1 \end{vmatrix}$$

$= \hat{i}(2-6) - \hat{j}(1-6) + \hat{k}(2-4)$

$= -4\hat{i} - 5\hat{j} - 2\hat{k}$

$\ell = \gamma(-4\hat{i} + 5\hat{j} - 2\hat{k})$

P is intersection of ℓ and ℓ_1

$-4\gamma = 1 + \lambda, 5\gamma = -11 + 2\lambda, -2\gamma = -7 + 3\lambda$

By solving these equation $\gamma = -1, P(4, -5, 2)$

Let Q $(-1 + 2\mu, 2\mu, 1 + \mu)$

$\overrightarrow{PQ} \cdot (2\hat{i} + 2\hat{j} + \hat{k}) = 0$

$-2 + 4\mu + 4\mu + 1 + \mu = 0$

$9\mu = 1$

$\mu = \frac{1}{9}$

$Q\left(\frac{-7}{9}, \frac{2}{9}, \frac{10}{9}\right)$

$9(\alpha + \beta + \gamma) = 9\left(\frac{-7}{9} + \frac{2}{9} + \frac{10}{9}\right)$

$= 5$