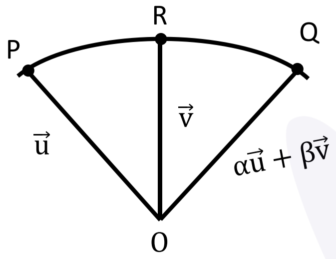
FINAL JEE–MAIN EXAMINATION – APRIL, 2023
Held On Monday 10th April, 2023
TIME : 09:00 AM to 12:00 PM

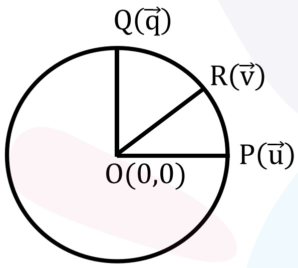
SECTION-A

1. An arc PQ of a circle subtends a right angle at its centre O. The mid point of the arc PQ is R. If $\overrightarrow{OP} = \vec{u}, \overrightarrow{OR} = \vec{v}$ and $\overrightarrow{OQ} = \alpha\vec{u} + \beta\vec{v}$, then α, β^2 are the roots of the equation :

(1) $3x^2 - 2x - 1 = 0$ (2) $3x^2 + 2x - 1 = 0$ (3) $x^2 - x - 2 = 0$ (4) $x^2 + x - 2 = 0$

Sol. **(3)**





Let $\overrightarrow{OP} = \vec{u} = \hat{i}$

$\overrightarrow{OQ} = \vec{q} = \hat{j}$

$\because$ R is the mid point of $\overrightarrow{PQ}$

Then $\overrightarrow{OR} = \vec{v} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Now

$\overrightarrow{OQ} = \alpha\vec{u} + \beta\vec{v}$

$\hat{j} = \alpha\hat{i} + \beta\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$

$\beta = \sqrt{2}, \alpha + \frac{\beta}{\sqrt{2}} = 0 \Rightarrow \alpha = -1$

Now equation

$x^2 - (\alpha + \beta^2)x + \alpha\beta^2 = 0$

$x^2 - (-1 + 2)x + (-1)(2) = 0$

$\boxed{x^2 - x - 2 = 0}$

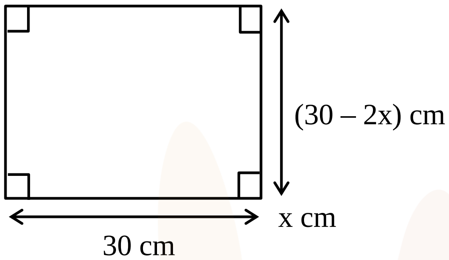
2. A square piece of tin of side 30 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. If the volume of the box is maximum, then its surface area (in cm^2) is equal to :

(1) 800 (2) 1025 (3) 900 (4) 675

Sol. **(1)**

Let the side of the square to be cut off be x cm.

Then, the length and breadth of the box will be (30 – 2x) cm each and the height of the box is x cm therefore,



The volume V(x) of the box is given by

$V(x) = x(30 - 2x)^2$

$\frac{dv}{dx} = (30 - 2x)^2 + 2x \times (30 - 2x)(-2)$

$0 = (30 - 2x)^2 - 4x(30 - 2x)$

$0 = (30 - 2x)[(30 - 2x) - 4x]$

$0 = (30 - 2x)(30 - 6x)$

$x = 15, 5$

$x \neq 15$ (Not possible)

$\{\because V = 0\}$

Surface area without top of the box = $\ell b + 2(bh + h\ell)$

$= (30 - 2x)(30 - 2x) + 2[(30 - 2x)x + (30 - 2x)x]$

$= [(30 - 2x)^2 + 4\{(30 - 2x)x\}$

$= [(30 - 10)^2 + 4(5)(30 - 10)]$

$= 400 + 400$

$= 800 \text{ cm}^2$

3. Let O be the origin and the position vector of the point P be $-\hat{i} - 2\hat{j} + 3\hat{k}$. If the position vectors of the A, B and C are $-2\hat{i} + \hat{j} - 3\hat{k}, 2\hat{i} + 4\hat{j} - 2\hat{k}$ and $-4\hat{i} + 2\hat{j} - \hat{k}$ respectively, then the projection of the vector $\overrightarrow{OP}$ on a vector perpendicular to the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is :

(1) $\frac{10}{3}$ (2) $\frac{8}{3}$ (3) $\frac{7}{3}$ (4) 3

Sol. **(4)**

Position vector of the point $P(-1,-2,3)$, $A(-2,1,-3)$ $B(2,4,-2)$, and $C(-4,2,-1)$

Then $\overrightarrow{OP} \cdot \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|(\overrightarrow{AB} \times \overrightarrow{AC})|}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & 1 \\ -2 & 1 & 2 \end{vmatrix}$$

$= \hat{i}(5) - \hat{j}(8 + 2) + \hat{k}(4 + 6)$

$= 5\hat{i} - 10\hat{j} + 10\hat{k}$

Now

$$\overrightarrow{OP}.\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|(\overrightarrow{AB}\times\overrightarrow{AC})|}=(-\hat{i}-2\hat{j}+3\hat{k}).\frac{(5\hat{i}-10\hat{j}+10\hat{k})}{\sqrt{(5)^2+(-10)^2+(10)^2}}$$

$$=\frac{-5+20+30}{\sqrt{25+100+100}}$$

$$=\frac{45}{\sqrt{225}}=\frac{45}{15}=3$$

4. If A is a 3 × 3 matrix and |A| = 2, then $\left|3\text{adj}(|3A|A^2)\right|$ is equal to :

(1) $3^{12}.6^{10}$ (2) $3^{11}.6^{10}$ (3) $3^{12}.6^{11}$ (4) $3^{10}.6^{11}$

Sol. **(2)**

Given |A|= **2**

Now, $|3\text{adj}(|3A|A^2)|$

$|3A| = 3^3.|A|$

$= 3^3.(2)$

Adj. $(|3A|A^2) = \text{adj}\{(3^3.2)A^2\}$

$= (2.3^3)^2(\text{adj }A)^2$

$= 2^2.3^6.(\text{adj }A)^2$

$|3\text{ adj}(|3A|A^2)| = |2^2.3.3^6(\text{adj }A)^2|$

$= (2^2.3^7)^3|\text{adj }A|^2$

$= 2^6.3^{21}(|A|^2)^2$

$= 2^6.3^{21}(2^2)^2$

$= 2^{10}.3^{21}$

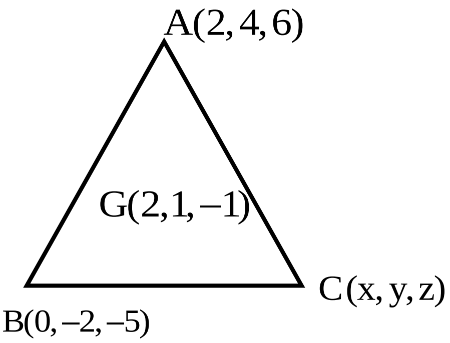
$= 2^{10}.3^{10}.3^{11}$

$|3\text{ adj}(|3A|A^2)| = 6^{10}.3^{11}$

5. Let two vertices of a triangle ABC be (2, 4, 6) and (0, –2, –5), and its centroid be (2, 1, –1). If the image of the third vertex in the plane x + 2y + 4z = 11 is (α, β, γ), then αβ + βγ + γα is equal to :

(1) 76 (2) 74 (3) 70 (4) 72

Sol. **(2)**



Given Two vertices of Triangle $A(2,4,6)$ and $B(0,-2,-5)$ and if centroid $G(2,1,-1)$

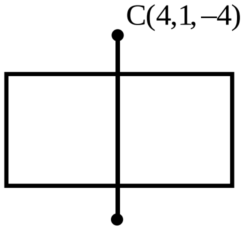
Let Third vertices be (x,y,z)

Now $\frac{2+0+x}{3}=2,\frac{4-2+y}{3}=1,\frac{6-5+z}{3}=-1$

$x=4,\ y=1,\ z=-1$

Third vertices $C(4,1,-4)$

Now, Image of vertices C(4,1,–4) in the given plane is D(α,β,γ)



Now

$$\frac{\alpha-4}{1}=\frac{\beta-1}{2}=\frac{\gamma+4}{4}=-2\frac{(4+2-16-11)}{1+4+16}$$

$$\frac{\alpha-4}{1}=\frac{\beta-1}{2}=\frac{\gamma+4}{4}=\frac{42}{21}\Rightarrow 2$$

$$\alpha=6,\ \beta=5,\ \ \gamma=4$$

Then $\alpha\beta+\beta\gamma+\gamma\alpha$

$=(6\times5)+(5\times4)+(4\times6)$

$=30+20+24$

$=74$

6. The negation of the statement :

$(p \vee q) \wedge (q \vee (\sim r))$ is

(1) $((\sim p) \vee r)) \wedge (\sim q)$

(2) $((\sim p) \vee (\sim q)) \wedge (\sim r)$

(3) $((\sim p) \vee (\sim q)) \vee (\sim r)$

(4) $(p \vee r) \wedge (\sim q)$

Sol. **(1)**

$(p\vee q)\wedge(q\vee(\sim r))$

$\sim\left[(p\vee q)\wedge(q\vee(\sim r))\right]$

$=\sim(p\vee q)\wedge(\sim q\wedge r)$

$=(\sim p\wedge\sim q)\vee(\sim q\wedge r)$

$=(\sim p\vee r)\wedge(\sim q)$

7. The shortest distance between the lines $\frac{x+2}{1}=\frac{y}{-2}=\frac{z-5}{2}$ and $\frac{x-4}{1}=\frac{y-1}{2}=\frac{z+3}{0}$ is :

(1) 8 (2) 7 (3) 6 (4) 9

Sol. **(4)**

$$\frac{x+2}{1}=\frac{y}{-2}=\frac{z-5}{2} \text{ and } \frac{x-4}{1}=\frac{y-1}{2}=\frac{z+3}{0}$$

$$\text{Shortest distance } (d) = \frac{\left\| \begin{matrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{matrix} \right\|}{\left\| \begin{matrix} \hat{i} & \hat{j} & k \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{matrix} \right\|}$$

$$= \frac{\left\| \begin{matrix} 4+2 & 1-0 & -3-5 \\ 1 & -2 & 2 \\ 1 & 2 & 0 \end{matrix} \right\|}{\left\| \begin{matrix} \hat{i} & \hat{j} & k \\ 1 & -2 & 2 \\ 1 & 2 & 0 \end{matrix} \right\|}$$

$$= \frac{\left| \begin{matrix} 6 & 1 & -8 \\ 1 & -2 & 2 \\ 1 & 2 & 0 \end{matrix} \right|}{\left| \hat{i}(-4) - \hat{j}(-2) + k(2+2) \right|}$$

$$= \frac{|-54|}{\left| -4\hat{i} + 2\hat{j} + 4k \right|}$$

$$= \frac{54}{\sqrt{16+4+16}}$$

$$= \frac{54}{6}$$

$$= 9$$

8. If the coefficient of x^7 in $\left(ax - \frac{1}{bx^2}\right)^{13}$ and the coefficient of x^{-5} in $\left(ax + \frac{1}{bx^2}\right)^{13}$ are equal, then a^4b^4 is equal to :

(1) 22 (2) 44 (3) 11 (4) 33

Sol. **(1)**

$$\left(ax - \frac{1}{bx^2}\right)^{13}$$

We have,

$T_{r+1} = {}^nC_r\,(p)^{n-r}\,(q)^r$

$$T_{r+1} = {}^{13}C_r\,(ax)^{13-r}\left(-\frac{1}{bx^2}\right)^r$$

$$= {}^{13}C_r\,(a)^{13-r}\left(-\frac{1}{b}\right)^r (x)^{13-r}.(x)^{-2r}$$

$$= {}^{13}C_r\,(a)^{13-r}\left(-\frac{1}{b}\right)^r (x)^{13-3r} \quad \ldots(1)$$

Coefficient of x^7

$\Rightarrow 13 - 3r = 7$

$r = 2$

r in equation (1)

$T_3 = {}^{13}C_2 (a)^{13-2} \left(-\frac{1}{b}\right)^2 (x)^{13-6}$

$= {}^{13}C_2 (a)^{11} \left(\frac{1}{b}\right)^2 (x)^7$

Coefficient of x^7 is ${}^{13}C_2 \frac{(a)^{11}}{b^2}$

Now, $\left(ax + \frac{1}{bx^2}\right)^{13}$

$T_{r+1} = {}^{13}C_r (ax)^{13-r} \left(\frac{1}{bx^2}\right)^r$

$= {}^{13}C_r (a)^{13-r} \left(\frac{1}{b}\right)^r (x)^{13-r}(x)^{-2r}$

$= {}^{13}C_r (a)^{13-r} \left(\frac{1}{b}\right)^r (x)^{13-3r}$ …(2)

Coefficient of x^{-5}

$\Rightarrow 13 - 3r = -5$

$r = 6$

r in equation

$T_7 = {}^{13}C_6 (a)^{13-6} \left(\frac{1}{b}\right)^6 (x)^{13-18}$

$T_7 = {}^{13}C_6 (a)^7 \left(\frac{1}{b}\right)^6 (x)^{-5}$

Coefficient of x^{-5} is ${}^{13}C_6 (a)^7 \left(\frac{1}{b}\right)^6$

ATQ

Coefficient of x^7 = coefficient of x^{-5}

$T_3 = T_7$

${}^{13}C_2 \left(\frac{a^{11}}{b^2}\right) = {}^{13}C_6 (a)^7 \left(\frac{1}{b}\right)^6$

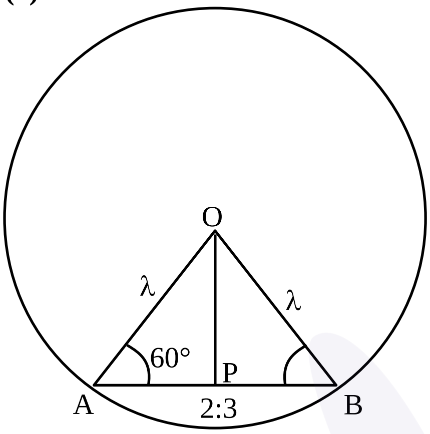
$a^4.b^4 = \frac{{}^{13}C_6}{{}^{13}C_2}$

$= \frac{13\times12\times11\times10\times9\times8\times1}{13\times12\times6\times5\times4\times3} = 22$

9. A line segment AB of length λ moves such that the points A and B remain on the periphery of a circle of radius λ. Then the locus of the point, that divides the line segment AB in the ratio 2 : 3, is a circle of radius :

(1) $\frac{2}{3}\lambda$ (2) $\frac{\sqrt{19}}{7}\lambda$ (3) $\frac{3}{5}\lambda$ (4) $\frac{\sqrt{19}}{5}\lambda$

Sol. **(4)**



Since OAB form equilateral Δ

$\therefore \angle OAP = 60^\circ$

$AP = \frac{2\lambda}{5}$

$\cos 60^\circ = \frac{OA^2 + AP^2 - OP^2}{2OA.AP}$

$\Rightarrow \frac{1}{2} = \frac{\lambda^2 + \frac{4\lambda^2}{25} - OP^2}{2\lambda\left(\frac{2\lambda}{5}\right)}$

$\Rightarrow \frac{2\lambda^2}{5} = \lambda^2 + \frac{4\lambda^2}{25} - OP^2$

$\Rightarrow OP^2 = \frac{19\lambda^2}{25}$

$\Rightarrow OP = \frac{\sqrt{19}}{5}\lambda$

Therefore, locus of point P is $\frac{\sqrt{19}}{5}\lambda$

10. For the system of linear equations

$2x - y + 3z = 5$

$3x + 2y - z = 7$

$4x + 5y + \alpha z = \beta,$

which of the following is NOT correct ?

(1) The system in inconsistent for $\alpha = -5$ and $\beta = 8$

(2) The system has infinitely many solutions for $\alpha = -6$ and $\beta = 9$

(3) The system has a unique solution for $\alpha \neq -5$ and $\beta = 8$

(4) The system has infinitely many solutions for $\alpha = -5$ and $\beta = 9$

Sol. **(2)**

Given

$2x - y + 3z = 5$

$3x + 2y - z = 7$

$4x + 5y + \alpha z = \beta$

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & \alpha \end{vmatrix} = 7\alpha + 35$$

$\Delta = 7(\alpha + 5)$

For unique solution $\Delta \neq 0$

$\alpha \neq -5$

For inconsistent & Infinite solution

$\Delta = 0$

$\alpha + 5 = 0 \Rightarrow \alpha = -5$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 3 \\ 7 & 2 & -1 \\ \beta & 5 & -5 \end{vmatrix} = -5(\beta - 9)$$

$$\Delta_2 = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 7 & -1 \\ 4 & \beta & -5 \end{vmatrix} = 11(\beta - 9)$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 5 \\ 3 & 2 & 7 \\ 4 & 5 & \beta \end{vmatrix}$$

$\Delta_3 = 7(\beta - 9)$

For Inconsistent system : -

At least one Δ_1, Δ_2 & Δ_3 is not zero $\alpha = -5$, $\beta = 8$ option (A) True

Infinite solution:

$\Delta_1 = \Delta_2 = \Delta_3 = 0$

From here $\beta - 9 = 0 \Rightarrow \beta = 9$

$\alpha = -5$ & option (D) True

$\beta = 9$

Unique solution

$\alpha \neq -5$, $\beta = 8 \rightarrow$ option (C) True

Option (B) False

For Infinitely many solution α must be -5.

11. Let the first term a and the common ratio r of a geometric progression be positive integers. If the sum of squares of its first three is 33033, then the sum of these terms is equal to :

(1) 210 (2) 220 (3) 231 (4) 241

Sol. **(3)**

Let a, ar , ar^2 be three terms of GP

Given : $a^2 + (ar)^2 + (ar^2)^2 = 33033$

$a^2(1 + r^2 + r^4) = 11^2.3.7.13$

$\Rightarrow a = 11$ and $1 + r^2 + r^4 = 3.7.13$

$\Rightarrow r^2(1 + r^2) = 273 - 1$

$\Rightarrow r^2(r^2 + 1) = 272 = 16 \times 17$

$\Rightarrow r^2 = 16$

$\therefore r = 4$ $[\because r > 0]$

Sum of three terms = $a + ar + ar^2 = a(1 + r + r^2)$

$= 11(1 + 4 + 16)$

$= 11 \times 21 = 231$

12. Let P be the point of intersection of the line $\frac{x+3}{3}=\frac{y+2}{1}=\frac{1-z}{2}$ and the plane x + y + z = 2. If the distance of the point P from the plane 3x – 4y + 12z = 32 is q, then q and 2q are the roots of the equation :

(1) $x^2 + 18x - 72 = 0$ (2) $x^2 + 18x + 72 = 0$ (3) $x^2 - 18x - 72 = 0$ (4) $x^2 - 18x + 72 = 0$

Sol. **(4)**

$\frac{x+3}{3} = \frac{y+2}{1} = \frac{1-z}{2} = \lambda$

$x = 3\lambda - 3,\ y = \lambda - 2,\ z = 1 - 2\lambda$

$P(3\lambda - 3, \lambda - 2, 1 - 2\lambda)$ will satisfy the equation of plane x + y + z = 2.

$3\lambda - 3 + \lambda - 2 + 1 - 2\lambda = 2$

$2\lambda - 4 = 2$

$\lambda = 3$

P(6, 1, – 5)

Perpendicular distance of P from plane 3x – 4y + 12 z – 32 = 0 is

$q = \left|\frac{3(6) - 4(1) + 12(-5) - 32}{\sqrt{9+16+144}}\right|$

q = 6.

2q = 12

Sum of roots = 6 + 12 = 18

Product of roots = 6(12) = 72

$\therefore$ Quadratic equation having q and 2q as roots is $x^2 - 18x + 72$.

13. Let f be a differentiable function such that $x^2 f(x) - x = 4\int_0^x t f(t)\,dt$, $f(1) = \frac{2}{3}$. Then 18 f (3) is equal to :

(1) 180 (2) 150 (3) 210 (4) 160

Sol. **(4)**

$x^2 f(x) - x = 4\int_0^x t f(t)\,dt$

Differentiate w.r.t. x

$x^2 f'(x) + 2x f(x) - 1 = 4x f(x)$

Let y = f(x)

$\Rightarrow x^2 \frac{dy}{dx} - 2xy - 1 = 0$

$\frac{dy}{dx} - \frac{2}{x} y = \frac{1}{x^2}$

$\text{I.F.} = e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}$

Its solution is

$\frac{y}{x^2} = \int \frac{1}{x^4}\,dx + C$

$\frac{y}{x^2} = \frac{-1}{3x^3} + C$

$\because f(1) = \frac{2}{3} \Rightarrow y(1) = \frac{2}{3}$

$\Rightarrow \frac{2}{3} = -\frac{1}{3} + C$

$\Rightarrow C = 1$

$\because y = -\frac{1}{3x} + x^2$

$f(x) = -\frac{1}{3x} + x^2$

$f(3) = -\frac{1}{9} + 9 = \frac{80}{9} \Rightarrow 18f(3) = 160$

14. Let N denote the sum of the numbers obtained when two dice are rolled. If the probability that $2^N < N!$ is $\frac{m}{n}$, where m and n are coprime, then 4m –3n equal to :

(1) 12 (2) 8 (3) 10 (4) 6

Sol. **(2)**

$2^N < N!$ is satisfied for $N \geq 4$

Required probability $P(N \geq 4) = 1 - P(N < 4)$

N = 1 (Not possible)

N = 2 (1, 1)

$\Rightarrow P(N = 2) = \frac{1}{36}$

N = 3 (1, 2), (2, 1)

$\Rightarrow P(N = 3) = \frac{2}{36}$

$P(N < 4) = \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$

$\therefore P(N \geq 4) = 1 - \frac{3}{36} = \frac{33}{36} = \frac{11}{12} = \frac{m}{n}$

$\Rightarrow m = 11, n = 12$

$\therefore 4m - 3n = 4(11) - 3(12) = 8$

15. If $I(x) = \int e^{\sin^2 x} (\cos x \sin 2x - \sin x)\, dx$ and I (0) = 1, then $I\left(\frac{\pi}{3}\right)$ is equal to :

(1) $e^{\frac{3}{4}}$ (2) $-e^{\frac{3}{4}}$ (3) $\frac{1}{2}e^{\frac{3}{4}}$ (4) $-\frac{1}{2}e^{\frac{3}{4}}$

Sol. **(3)**

$I = \int \underbrace{e^{\sin^2 x} \sin 2x}_{II} \underbrace{\cos x dx}_{I} - \int e^{\sin^2 x} \sin x dx$

$= \cos x \int e^{\sin^2 x} \sin 2x dx - \int \left((-\sin x) \int e^{\sin^2 x} \sin 2x dx\right) dx - \int e^{\sin^2 x} \sin x dx$

$\sin^2 x = t$

$\sin 2x\, dx = dt$

$= \cos x \int e^t dt + \int (\sin x \int e^t dt) dx - \int e^{\sin^2 x} \sin x dx$

$= e^{\sin^2 x} \cos x + \int e^{\sin^2 x} \sin x dx - \int e^{\sin^2 x} \sin x dx$

$I = e^{\sin^2 x}\cos x + C$

I (0) = 1

$\Rightarrow 1 = 1 + C$

$\Rightarrow C = 0$

$\therefore\ I = e^{\sin^2 x}\cos x$

$I\left(\frac{\pi}{3}\right) = e^{\sin^2\frac{\pi}{3}}\cos\frac{\pi}{3}$

$= \frac{e^{\frac{3}{4}}}{2}$

16. $96\cos\frac{\pi}{33}\cos\frac{2\pi}{33}\cos\frac{4\pi}{33}\cos\frac{8\pi}{33}\cos\frac{16\pi}{33}$ is equal to :

(1) 4 (2) 2 (3) 3 (4) 1

Sol. **(3)**

$96\cos\frac{\pi}{33}\cos\frac{2\pi}{33}\cos\frac{2^2\pi}{33}\cos\frac{2^3\pi}{33}\cos\frac{2^4\pi}{33}$

$\because \cos A\cos 2A\cos 2^2A \ldots \cos 2^{n-1}A = \frac{\sin(2^n A)}{2^n \sin A}$

Here $A = \frac{\pi}{33}$, n = 5

$= \frac{96\sin\left(2^5\frac{\pi}{33}\right)}{2^5\sin\left(\frac{\pi}{33}\right)}$

$= \frac{96\sin\left(\frac{32\pi}{33}\right)}{32\sin\left(\frac{\pi}{33}\right)}$

$= \frac{3\sin\left(\pi-\frac{\pi}{33}\right)}{\sin\left(\frac{\pi}{33}\right)} = 3$

17. Let the complex number z = x + iy be such that $\frac{2z-3i}{2z+i}$ is purely imaginary. If $x + y^2 = 0$, then $y^4 + y^2 - y$ is equal to :

(1) $\frac{3}{2}$ (2) $\frac{2}{3}$ (3) $\frac{4}{3}$ (4) $\frac{3}{4}$

Sol. **(4)**

z = x + iy

$\frac{(2z-3i)}{2z+i}$ = purely imaginary

Means $\text{Re}\left(\frac{2z-3i}{2z+i}\right) = 0$

$$\Rightarrow \frac{(2z-3i)}{(2z+i)} = \frac{2(x+iy)-3i}{2(x+iy)+i}$$

$$= \frac{2x+2yi-3i}{2x+i2y+i}$$

$$= \frac{2x+i(2y-3)}{2x+i(2y+1)} \times \frac{2x-i(2y+1)}{2x-i(2y+1)}$$

$$\text{Re}\left[\frac{2z-3i}{2z+i}\right] = \frac{4x^2+(2y-3)(2y+1)}{4x^2+(2y+1)^2} = 0$$

$\Rightarrow 4x^2 + (2y-3)(2y+1) = 0$

$\Rightarrow 4x^2 + [4y^2 + 2y - 6y - 3] = 0$

$\because x + y^2 = 0 \Rightarrow x = -y^2$

$\Rightarrow 4(-y^2)^2 + 4y^2 - 4y - 3 = 0$

$\Rightarrow 4y^4 + 4y^2 - 4y = 3$

$\Rightarrow y^4 + y^2 - y = \frac{3}{4}$

Therefore, correct answer is option (4).

18. If $f(x) = \frac{(\tan 1°)x + \log_e(123)}{x\log_e(1234) - (\tan 1°)}$, $x > 0$, then the least value of $f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right)$ is :

(1) 2 (2) 4 (3) 8 (4) 0

Sol. **(2)**

$$f(x) = \frac{(\tan 1)x + \log 123}{x\log 1234 - \tan 1}$$

Let $A = \tan 1,\quad B = \log 123,\quad C = \log 1234$

$$f(x) = \frac{Ax+B}{xC-A}$$

$$f(f(x)) = \frac{A\left(\frac{Ax+B}{xC-A}\right)+B}{C\left(\frac{Ax+B}{CX-A}\right)-A}$$

$$= \frac{A^2x+AB+xBC-AB}{ACx+BC-ACx+A^2}$$

$$= \frac{x(A^2+BC)}{(A^2+BC)} = x$$

$$f(f(x)) = x$$

$$f\left(f\left(\frac{4}{x}\right)\right) = \frac{4}{x}$$

$$f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right)$$

$\text{AM} \geq \text{GM}$

$$x + \frac{4}{x} \geq 4$$

19. The slope of tangent at any point (x, y) on a curve y = y(x) is $\frac{x^2+y^2}{2xy}, x>0$. If y (2) = 0, then a value of y (8) is :

(1) $4\sqrt{3}$ (2) $-4\sqrt{2}$ (3) $-2\sqrt{3}$ (4) $2\sqrt{3}$

Sol. **(1)**

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

y = vx

y(2) = 0

y(8) = ?

$\frac{dv}{dx} = v + x\frac{dv}{dx}$

$v + \frac{xdv}{dx} = \frac{x^2+v^2x^2}{2vx^2}$

$x.\frac{dv}{dx} = \left(\frac{v^2+1}{2v} - v\right)$

$\frac{2vdv}{(1-v^2)} = \frac{dx}{x}$

$-\ln(1-v^2) = \ln x + C$

$\ln x + \ln(1-v^2) = C$

$\ln\left[x\left(1-\frac{y^2}{x^2}\right)\right] = C$

$\ln\left[\left(\frac{x^2-y^2}{x}\right)\right] = C$

$x^2 - y^2 = cx$

y(2) = 0 at x = 2, y = 0

$4 = 2C \Rightarrow C = 2$

$x^2 - y^2 = 2x$

Hence, at x = 8

$64 - y^2 = 16$

$y = \sqrt{48} = 4\sqrt{3}$

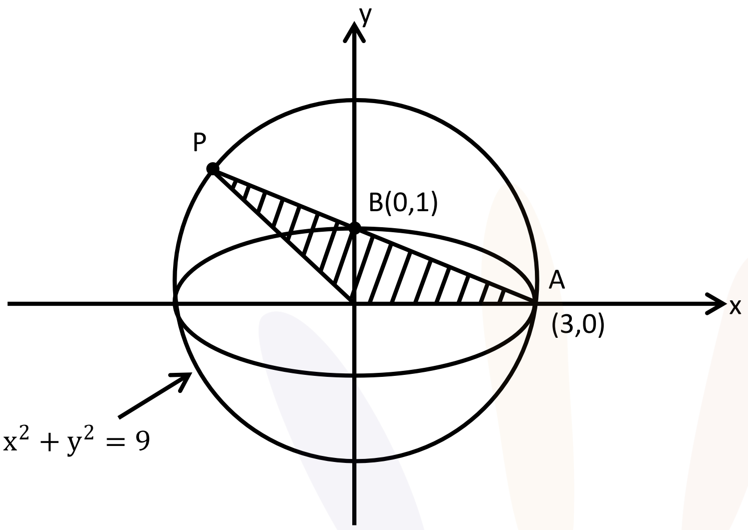
$y(8) = 4\sqrt{3}$

Option (1)

20. Let the ellipse E : $x^2 + 9y^2 = 9$ intersect the positive x-and y-axes at the points A and B respectively. Let the major axis of E be a diameter of the circle C. Let the line passing through A and B meet the circle C at the point P. If the area of the triangle with vertices A, P and the origin O is $\frac{m}{n}$, where m and n are coprime, then m – n is equal to :

(1) 16 (2) 15 (3) 18 (4) 17

Sol. **(4)**



Equation of line AB or AP is

$$\frac{x}{3}+\frac{y}{1}=1$$

$$x+3y=3$$

$$x=(3-3y)$$

Intersection point of line AP & circle is $P(x_0, y_0)$

$$x^2+y^2=9 \Rightarrow (3-3y)^2+y^2=9$$

$$\Rightarrow 3^2(1+y^2-2y)+y^2=9$$

$$\Rightarrow 5y^2-9y=0 \Rightarrow y(5y-9)=0$$

$$\Rightarrow y=9/5$$

Hence $x=3(1-y)=3\left(1-\frac{9}{5}\right)=3\left(\frac{-4}{5}\right)$

$$x=\frac{-12}{5}$$

$$P(x_0, y_0)=\left(\frac{-12}{5},\frac{9}{5}\right)$$

Area of ΔAOP is $=\frac{1}{2}\times OA\times \text{height}$

Height $=9/5$, $OA=3$

$$=\frac{1}{2}\times 3\times\frac{9}{5}=\frac{27}{10}=\frac{m}{n}$$

Compare both side $m=27$, $n=10$ $\Rightarrow m-n=17$

Therefore, correct answer is option-D

SECTION-B

21. Some couples participated in a mixed doubles badminton tournament. If the number of matches played, so that no couple in a match, is 840, then the total numbers of persons, who participated in the tournament, is ____.

Sol. 16

Let number of couples = n

$\therefore {}^{n}C_2 \times {}^{n-2}C_2 \times 2 = 840$

$\Rightarrow n(n-1)(n-2)(n-3) = 840 \times 2$

$= 21 \times 40 \times 2$

$= 7 \times 3 \times 8 \times 5 \times 2$

$n(n-1)(n-2)(n-3) = 8 \times 7 \times 6 \times 5$

$\therefore n = 8$

Hence, number of persons = 16.

22. The number of elements in the set $\{n \in Z : |n^2 - 10n + 19| < 6\}$ is _____.

Sol. 6

$-6 < n^2 - 10n + 19 < 6$

$\Rightarrow n^2 - 10n + 25 > 0$ and $\quad n^2 - 10n + 13 < 0$

$(n-5)^2 > 0 \quad 5 - 3\sqrt{2} < n < 5 + 3\sqrt{2}$

$N \in Z - \{5\} \quad n = \{2, 3, 4, 5, 6, 7, 8\}$

…(i) …(ii)

From (i) $\cap$ (ii)

$N = \{2, 3, 4, 5, 6, 8,\}$

Number of values of n = 6

23. The number of permutations of the digits 1, 2, 3, …., 7 without repetition, which neither contain the string 153 nor the string 2467, is _____.

Sol. 4898

Numbers are 1, 2, 3, 4, 5, 6, 7

Numbers having string (154) = (154), 2, 3, 6, 7 = 5!

Numbers having string (2467) = (2467), 1,3, 5 = 4!

Number having string (154) and (2467)

= (154), (2467) = 2!

Now n $(154 \cup 2467) = 5! + 4! - 2!$

$= 120 + 24 - 2 = 142$

Again total numbers = 7! = 5040

Now required numbers = n (neither 154 nor 2467)

$= 5040 - 142$

$= 4898$

24. Let f: $(-2, 2) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x[x] & , -2 < x < 0 \\ (x-1)[x], & 0 \le x < 2 \end{cases}$$

where [x] denotes the greatest integer function. If m and n respectively are the number of points in (–2, 2) at which y = |f(x)| is not continuous and not differentiable, then m + n is equal to ______.

Sol. **4**

$$f(x) = \begin{cases} -2x, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ x-1, & 1 \le x < 2 \end{cases}$$

Clearly f(x) is discontinuous at x = –1 also non differentiable.

$\therefore$ m = 1

Now for differentiability

$$f'(x) = \begin{cases} -2 & -2 < x < -1 \\ -1 & -1 < x < 0 \\ 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

Clearly f(x) is non-differentiable at x = –1, 0, 1

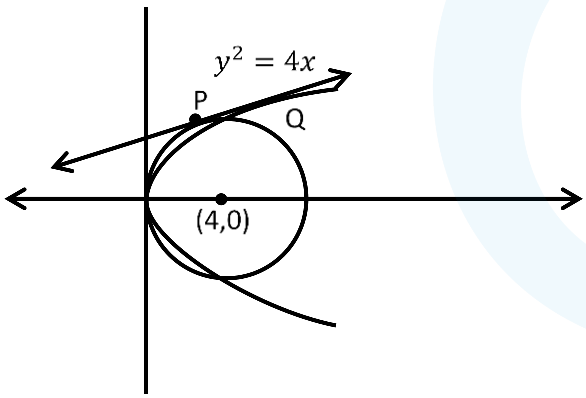
Also, $|f(x)|$ remains same.

$\therefore$ n = 3

$\therefore$ m + n = 4

25. Let a common tangent to the curves $y^2 = 4x$ and $(x-4)^2 + y^2 = 16$ touch the curves at the points P and Q. Then $(PQ)^2$ is equal to ____ :

Sol. **32**



$y^2 = 4x$

$(x-4)^2 + y^2 = 16$

Let equation of tangent of parabola

$y = mx + 1/m$ ..(1)

Now equation 1 also touches the circle

$$\therefore \left|\frac{4m + 1/m}{\sqrt{1+m^2}}\right| = 4$$

$(4m + 1/m)^2 = 16 + 16m^2$

$16m^4 + 8m^2 + 1 = 16m^2 + 16m^4$

$8m^2 = 1$

$\boxed{m^2 = 1/8}$ $\{m^4 = 0\}(m \to \infty)$

For distinet points consider only $m^2 = 1/8$.

Point of contact of parabola

$P(8, 4\sqrt{2})$

$\therefore PQ = \sqrt{S_1} \Rightarrow (PQ)^2 = S_1$

$= 16 + 32 - 16 = 32$

26. If the mean of the frequency distribution

Class :	0-10	10-20	20-30	30-40	40-50
Frequency :	2	3	x	5	4

is 28, then its variance is ______.

Sol. **151**

C.I.	f	x	$f_i x_i$	x_i^2
0-10	2	5	10	25
10-20	3	15	45	225
20-30	x	25	25x	625
30-40	5	35	175	1225
40-50	4	45	180	2025

$\bar{x} = \dfrac{\Sigma f_i x_i}{N}$

$28 = \dfrac{10 + 45 + 25x + 175 + 130}{14 + x}$

$28 \times 14 + 28x = 410 + 25x$

$\Rightarrow 3x = 410 - 392$

$\Rightarrow x = \dfrac{18}{3} = 6$

$\therefore$ Variance $= \dfrac{1}{N} \Sigma f_i x_i^2 - (\bar{x})^2$

$= \dfrac{1}{20} 18700 - (28)^2$

$= 935 - 784 = 151$

27. The coefficient of x^7 in $(1 - x + 2x^3)^{10}$ is ______.

Sol. **960**

$(1 - x + 2x^3)^{10}$

a	b	c
3	7	0
5	4	1
7	1	2

$T_n = \dfrac{10!}{a!b!c!}(-2x)^b (x^3)^c$

$= \dfrac{10!}{a!b!c!} (-2)^b x^{b+3c}$

$\Rightarrow b + 3c = 7, a + b + c = 10$

$\therefore$ Coefficient of $x^7 = \frac{10!}{3!7!0!}(-1)^7 + \frac{10!}{5!4!1!}(-1)^4$ **(2)**

$+ \frac{10!}{7!1!2!}(-1)^1 (2)^2$

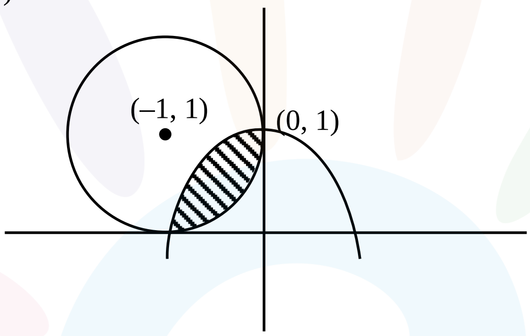
$= -120 + 2520 - 1440 = 960$

28. Let y = p(x) be the parabola passing through the points (–1, 0), (0, 1) and (1, 0). If the area of the region $\{(x, y) : (x+1)^2 + (y-1)^2 \leq 1, y \leq p(x)\}$ is A, then $12(\pi - 4A)$ is equal to _____:

Sol. **16**

There can be infinitely many parabolas through given points.

Let parabola $x^2 = -4a(y-1)$



Passes through (1, 0)

$\therefore\ b = -4a(-1) \Rightarrow a = \frac{1}{4}$

$\therefore\ x^2 = -(y-1)$

Now area covered by parabola = $\int_{-1}^{0} (1 - x^2)dx$

$= \left(x - \frac{x^3}{3}\right)_1^b = (0 - 0) - \left\{-1 + \frac{1}{3}\right\}$

$= \frac{2}{3}$

Required Area = Area of sector – {Area of square – Area covered by Parabola}

$= \frac{\pi}{4} - \left\{1 - \frac{2}{3}\right\}$

$= \frac{\pi}{4} - \frac{1}{3}$

$\therefore\ 12(\pi - 4A) = 12\left[\pi - 4\left(\frac{\pi}{4} - \frac{1}{3}\right)\right]$

$= 12\left[\pi - \pi + \frac{4}{3}\right]$

$= 16$

29. Let a, b, c be three distinct positive real numbers such that $(2a)^{\log_e a} = (bc)^{\log_e b}$ and $b^{\log_e 2} = a^{\log_e c}$. Then 6a + 5bc is equal to _____.

Sol. **Bouns**

$(2a)^{\ln a} = (bc)^{\ln b} \quad 2a > 0, bc > 0$

$\ln a(\ln 2 + \ln a) = \ln b(\ln b + \ln c)$

$\ln 2 \cdot \ln b = \ln c \cdot \ln a$

$\ln 2 = \alpha, \ln a = x, \ln b = y, \ln c = z$

$\alpha y = xz$

$x(\alpha + x) = y(y + z)$

$\alpha = \frac{xz}{y}$

$x\left(\frac{xz}{y} + x\right) = y(y + z)$

$x^2(z + y) = y^2(y + z)$

$y + z = 0$ or $x^2 = y^2 \Rightarrow x = -y$

$bc = 1$ or $ab = 1$

bc = 1 or ab = 1

(1) if bc = 1 ⇒ $(2a)^{\ln a} = 1$ → a = 1, a = 1/2

$(a, b, c) = \left(\frac{1}{2}, \lambda, \frac{1}{\lambda}\right), \lambda \neq 1, 2, \frac{1}{2}$

then

6a + 5bc = 3 + 5 = 8

$(II)(a, b, c) = \left(\lambda, \frac{1}{\lambda}, \frac{1}{2}\right), \lambda \neq 1, 2, \frac{1}{2}$

In this situation infinite answer are possible

So, Bonus.

30. The sum of all those terms, of the arithmetic progression 3, 8, 13, …., 373, which are not divisible by 3, is equal to _____.

Sol. **9525**

$A.P: 3, 8, 13.....373$

$T_n = a + (n-1)d$

$373 = 3 + (n-1)5$

$\Rightarrow n = \frac{370}{5}$

$\Rightarrow \boxed{n = 75}$

Now Sum $= \frac{n}{2}[a+l]$

$= \frac{75}{2}[3+373] = 14100$

Now numbers divisible by 3 are,

$3, 18, 33 .... 363$

$363 = 3 + (k-1)15$

$\Rightarrow k - 1 = \frac{360}{15} = 24 \Rightarrow \boxed{k = 25}$

Now, sum $= \frac{25}{2}(3+363) = 4575\,s$

$\therefore$ req. sum $= 14100 - 4575$

$= 9525$