FINAL JEE–MAIN EXAMINATION – APRIL, 2023
Held On Saturday 08th April, 2023
TIME : 03:00 PM to 06:00 PM

SECTION - A

1. Let A = $\left\{\theta \in (0,2\pi) : \frac{1+2i\sin\theta}{1-i\sin\theta} \text{ is purely imaginary}\right\}$. Then the sum of the elements in A is.

(1) π (2) 3π (3) 4π (4) 2π

Sol. **(3)**

$$z = \frac{1+2i\sin\theta}{1-i\sin\theta} \times \frac{1+i\sin\theta}{1+i\sin\theta}$$

$$z = \frac{1-2\sin^2\theta + i(3\sin\theta)}{1+\sin^2\theta}$$

Re(z) = 0

$$\frac{1-2\sin^2\theta}{1+\sin^2\theta} = 0$$

$$\sin\theta = \frac{\pm 1}{\sqrt{2}}$$

$$A = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$

sum = 4π (Option 3)

2. Let P be the plane passing through the line $\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+5}{7}$ and the point (2, 4, –3). If the image of the point (-1, 3, 4) in the plane P is (α, β, γ) then $\alpha + \beta + \gamma$ is equal to

(1) 12 (2) 9 (3) 10 (4) 11

Sol. **(3)**

Equation of plane is given by

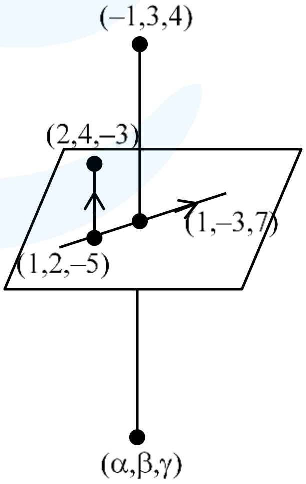
$$\begin{vmatrix} x-1 & y-2 & z+5 \\ 1 & 2 & 2 \\ 1 & -3 & 7 \end{vmatrix} = 0$$

$4x - y - z = 7$

$$\frac{\alpha+1}{4} = \frac{\beta-3}{-1} = \frac{\gamma-4}{-1} = \frac{-2(-4-3-4-7)}{16+1+1} = 2$$

$\alpha = 7, \beta = 1, \gamma = 2$

$\alpha + \beta + \gamma = 10$ (Option 3)



3. If $A = \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix}$, $A^{-1} = \alpha A + \beta I$ and $\alpha + \beta = -2$, then $4\alpha^2 + \beta^2 + \lambda^2$ is equal to :

(1) 14 (2) 12 (3) 19 (4) 10

Sol. **(1)**

$$|A - xI| = 0 \Rightarrow \begin{vmatrix} 1-x & 5 \\ \lambda & 10-x \end{vmatrix} = 0 \Rightarrow x^2 - 11x + 10 - 5\lambda = 0$$

$$\Rightarrow (10-5\lambda)A^{-1} = -A + 11I$$

$$\therefore \alpha = \frac{-1}{10-5\lambda} \quad \text{and} \quad \beta = \frac{+11}{10-5\lambda}$$

$$\alpha+\beta=-2 \Rightarrow \frac{10}{10-5\lambda}=-2 \Rightarrow 10\text{-}5\lambda=-5 \Rightarrow \lambda=3$$

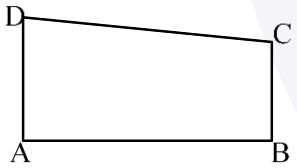
$$\therefore \alpha=\frac{1}{5} \qquad \& \qquad \beta=\frac{-11}{5}$$

$$\therefore 4a^2+\beta^2+\lambda^2=\frac{4}{25}+\frac{121}{25}+3^2=14 \text{ Ans.}$$

4. The area of the quadrilateral ABCD with vertices A(2,1,1), B (1,2, 5), C(–2,–3, 5) and D (1, -6, -7) is equal to

(1) 54 (2) $9\sqrt{38}$ (3) 48 (4) $8\sqrt{38}$

Sol. **(4)**



Vector Area = $\vec{v}$

$$=\frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}+\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$=\frac{1}{2}\left(\overrightarrow{AB}-\overrightarrow{AD}\right)\times\overrightarrow{AC}$$

$$\begin{pmatrix}\overrightarrow{AB}=-\hat{i}+\hat{j}+4\hat{k}\\ \overrightarrow{AD}=-\hat{i}-7\hat{j}-8\hat{k}\\ \overrightarrow{AC}=-4\hat{i}-4\hat{j}+4\hat{k}\end{pmatrix}$$

$$=\frac{1}{2}\left(8\hat{j}+12\hat{k}\right)\times(-4)\left(\hat{i}+\hat{j}-\hat{k}\right)$$

$$=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 8 & 12\\ 1 & 1 & -1\end{vmatrix}$$

$$= (-2)\left(-20\hat{i}+12\hat{j}-8\hat{k}\right)$$

$$=8\left(5\hat{i}-3\hat{j}+2\hat{k}\right)$$

$$\therefore \text{ Area } = |\vec{v}|=8\sqrt{25+9+4}=8\sqrt{38} \text{ Ans.}$$

5. $25^{190}-19^{190}-8^{190}+2^{190}$ is divisible by

(1) 34 but not by 14 (2) 14 but not by 34 (3) Both 14 and 34 (4) Neither 14 nor 34

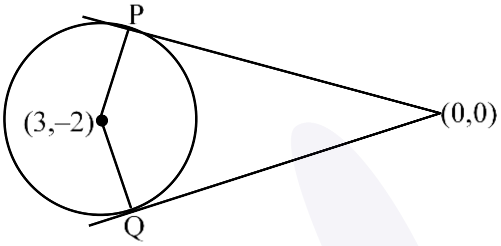
Sol. **(1)**

$25^{190}-8^{190}$ is divisible by 25 – 8 = 17
$19^{190}-2^{190}$ is divisible by 19 – 2 = 17
$25^{190}-19^{190}$ is divisible by 25 – 19 = 6
$8^{190}-2^{190}$ is divisible by 8 – 2 = 6
L.C.M. of 1746 = 34
∴ divisible by 34 but not by 14

6. Let O be the origin and OP and OQ be the tangents to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$ at the points P and Q on it. If the circumcircle of the triangle OPQ passes through the point $\left(\alpha, \frac{1}{2}\right)$, then a value of α is.

(1) $-\frac{1}{2}$ (2) $\frac{5}{2}$ (3) 1 (4) $\frac{3}{2}$

Sol. **(2)**



Circumcircle of ΔOPQ
$(x - 0)(x-3) + (y-0)(y + 2) = 0$
$x^2 + y^2 - 3x + 2y = 0$

passes through $\left(\alpha, \frac{1}{2}\right)$

$\therefore \alpha^2 + \frac{1}{4} - 3\alpha + 1 = 0$

$\Rightarrow \alpha^2 - 3\alpha + \frac{5}{4} = 0 \Rightarrow 4\alpha^2 - 12\alpha + 5 = 0$

$\Rightarrow 4\alpha^2 - 10\alpha - 2\alpha + 5 = 0$

$(2\alpha - 1)(2\alpha - 5) = 0 \ \therefore \alpha = \frac{1}{2}, \frac{5}{2}$ Ans.

7. Let a_n be the n^{th} term of the series 5 + 8 + 14 + 23 + 35 + 50 +.... and $Sn = \sum_{k=1}^{n} a_K$. Then $S_{30} - a_{40}$ is equal to

(1) 11260 (2) 11280 (3) 11290 (4) 11310

Sol. **(3)**

$S_n = 5 + 8 + 14 + 23 + 35 + 50 + ...+ a_n$
$S_n = \quad 5 + 8 + 14 + 23 + 35 + ...+ a_n$

$O = 5 + 3 + 6 + 9 + 12 + 15 + ... - a_n$
$a_n = 5 + (3 + 6 + 9 + ... (n-1) \text{ terms})$

$$a_n = \frac{3n^2 - 3n + 10}{2}$$

$$a_{40} = \frac{3(40)^2 - 3(40) + 10}{2} = 2345$$

$$S_{30} = \frac{3\sum_{n=1}^{30} n^2 - 3\sum_{n=1}^{30} n + 10\sum_{n=1}^{30} 1}{2}$$

$$= \frac{\frac{3 \times 30 \times 31 \times 61}{6} - \frac{3 \times 30 \times 31}{2} + 10 \times 30}{2}$$

$S_{30} = 13635$
$S_{30} - a_{40} = 13635 - 2345$
$= 11290$ (Option (3))

8. If α > β > 0 are the roots of the equation $ax^2 + bx + 1 = 0$, and $\lim\limits_{x\to\frac{1}{\alpha}}\left(\frac{1-\cos(x^2+bx+a)}{2(1-ax)^2}\right)^{\frac{1}{2}} = \frac{1}{k}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)$, then k is equal to

(1) β (2) 2α (3) 2 β (4) α

Sol. **(2)**

$\therefore ax^2 + bx + 1 = a(x-\alpha)(x-\beta) \ \therefore \alpha\beta = \frac{1}{a}$

$\therefore\ x^2 + bx + a = a(1-\alpha x)(1-\beta x)$

$$\therefore \lim_{x\to\frac{1}{\alpha}}\left\{\frac{1-\cos\left(x^2+bx+a\right)}{2(1-\alpha x)^2}\right\}^{\frac{1}{2}} = \lim_{x\to\frac{1}{2}}\left\{\frac{1-\cos a(1-\alpha x)(1-\beta x)}{2\{a(1-\alpha x)(1-\beta x)\}^2}\cdot a^2(1-\beta x)^2\right\}^{\frac{1}{2}}$$

$$=\left[\frac{1}{2}\cdot\frac{1}{2}a^2\left(1-\frac{\beta}{\alpha}\right)^2\right]^{\frac{1}{2}}$$

$$=\frac{1}{2}\frac{1}{\alpha\beta}\left(1-\frac{\beta}{\alpha}\right)=\frac{1}{2}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)$$

$$=\frac{1}{2\alpha}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)=\frac{1}{k}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)$$

$\therefore k = 2\alpha$ Ans.

9. If the number of words, with or without meaning, which can be made using all the letters of the word MATHEMATICS in which C and S do not come together, is (6!)k, is equal to

(1) 1890 (2) 945 (3) 2835 (4) 5670

Sol. **(4)**

$M_2A_2T_2HEICS$

= total words – when C & S are together

$$\frac{\lfloor\underline{11}}{\lfloor\underline{2}\,\lfloor\underline{2}\,\lfloor\underline{2}}-\frac{\lfloor\underline{10}}{\lfloor\underline{2}\,\lfloor\underline{2}\,\lfloor\underline{2}}\times\lfloor\underline{2}$$

$$\frac{\lfloor\underline{10}}{\lfloor\underline{2}\,\lfloor\underline{2}\,\lfloor\underline{2}}\times 9$$

$$=\frac{9\times10\times9\times8\times7}{8}\lfloor\underline{6}$$

$= 5670\ \lfloor\underline{6}$

k = 5670 (Option 4)

10. Let S be the set of all values of θ $\in[-\pi,\pi]$ for which the system of linear equations

$x+y+\sqrt{3}z=0$

$-x+(\tan\theta)y+\sqrt{7}z=0$

$x+y+(\tan\theta)z=0$

has non-trivial solution. Then $\frac{120}{\pi}\sum\limits_{\theta\in S}\theta$ is equal to

(1) 20 (2) 40 (3) 30 (4) 10

Sol. **(1)**

For non trivial solutions

D = 0

$$\begin{vmatrix} 1 & 1 & \sqrt{3} \\ -1 & \tan\theta & \sqrt{7} \\ 1 & 1 & \tan\theta \end{vmatrix} = 0$$

$$\tan^2\theta - \left(\sqrt{3}-1\right) - \sqrt{3} = 0$$

$$\tan\theta = \sqrt{3}, -1$$

$$\theta = \left\{\frac{\pi}{3}, \frac{-2\pi}{3}, \frac{-\pi}{4}, \frac{3\pi}{4}\right\}$$

$$\frac{120}{\pi}(\Sigma\theta) = \frac{120}{\pi} \times \frac{\pi}{6} = 20 \ \text{(Option 1)}$$

11. For a, b ∈ Z and $|a - b| \le 10$, let the angle between the plane P : ax + y – z = b and the line l : x –1 = a – y = z +1 be $\cos^{-1}\left(\frac{1}{3}\right)$. If the distance of the point (6, –6, 4) from the plane P is $3\sqrt{6}$, then $a^4 + b^2$ is equal to

(1) 85 (2) 48 (3) 25 (4) 32

Sol. **(4)**

$$\theta = \cos^{-1}\frac{1}{3} \therefore \sin\theta = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

$$\sin\theta = \frac{a\cdot 1 + 1(-1) + (-1)\cdot 1}{\sqrt{a^2+1+1}\cdot\sqrt{3}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \{3(a-2)\}^2 = 24\left(a^2+2\right)$$

$$\Rightarrow 3\left(a^2 - 4a + 4\right) = 8a^2 + 16$$

$$\Rightarrow 5a^2 + 12a + 4 = 0$$

$$\Rightarrow 5a^2 + 10a + 2a + 4 = 0$$

$$\therefore a = -2, \frac{-2}{5} \because a \in z$$

$$\therefore a = -2$$

Distance of (6, –6, 4) from

–2x + y – z – b = 0 is $3\sqrt{6}$

$$\therefore \left|\frac{-12-6-4-b}{\sqrt{4+1+1}}\right| = 3\sqrt{6}$$

$$\Rightarrow |b+22| = 18 \therefore b = -40, -4$$

$$\because |a-b| \le 10$$

$$\therefore b = -4$$

$$\therefore a^4 + b^2$$

= 32 Ans.

12. Let the vectors $\vec{u}_1 = \hat{i}+\hat{j}+a\hat{k}, \vec{\mu}_2 = \hat{i}+b\hat{j}+\hat{k}$ and $\vec{u}_3 = c\hat{i}+\hat{j}+\hat{k}$ be coplanar. If the vectors $\vec{v}_1 = (a+b)\hat{i}+c\hat{j}+c\hat{k}, \vec{v}_2 = a\hat{i}+(b+c)\hat{j}+a\hat{k}$ and $\vec{v}_3 = b\hat{i}+b\hat{j}+(c+a)\hat{k}$ are also coplanar, then 6 (a + b + c) is equal to

(1) 4 (2) 12 (3) 6 (4) 0

Sol. **(2)**

$$[\vec{u}_1\ \vec{u}_2\ \vec{u}_3] = 0 \quad \therefore \begin{vmatrix} 1 & 1 & a \\ 1 & b & 1 \\ c & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow b - 1 + c - 1 + a\,(1 - bc) = 0$

$\therefore\ abc = a + b + c - 2$

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = 0 \quad \therefore \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 0$$

$$R_3 \to R_3 - (R_1 + R_2) \Rightarrow \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ -2a & -2c & 0 \end{vmatrix} = 0$$

$\Rightarrow -2a\,(ac - bc - c^2) + 2c(a^2 + ab - ac) = 0$

$\Rightarrow -2a^2c + 2abc + 2ac^2 + 2a^2c + 2abc - 2ac^2 = 0$

$\Rightarrow 4abc = 0\ \therefore\ abc = 0$

$\therefore\ a + b + c = 2 \quad \therefore 6(a+b+c) = 12$ Ans.

13. The absolute difference of the coefficients of x^{10} and x^7 in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{11}$ is equal to

(1) $10^3 - 10$ (2) $11^3 - 11$ (3) $12^3 - 12$ (4) $13^3 - 13$

Sol. **(3)**

$T_{r+1} = {}^{11}C_r(2x^2)^{11-r}\left(\frac{1}{2x}\right)^r$

$= {}^{11}C_r\, 2^{11-2r}\, x^{22-3r}$

$22 - 3r = 10$ and $22 - 3r = 7$

$r = 4$ and $r = 5$

Coefficient of $x^{10} = {}^{11}C_4 \cdot 2^3$

Coefficient of $x^7 = {}^{11}C_5 \cdot 2^1$

difference $= {}^{11}C_4 \cdot 2^3 - {}^{11}C_5 \cdot 2$

$= \frac{11\times10\times9\times8}{24}\times8 - \frac{11\times10\times9\times8\times7}{120}\times2$

$= 11 \times 10 \times 3 \times 8 - 11 \times 3 \times 4 \times 7$

$= 11 \times 3\times 4 \times (20 - 7)$

$= 11 \times 12 \times 13$

$= 12\,(12 - 1)\,(12 + 1)$

$= 12\,(12^2 - 1)$

$= 12^3 - 12$ (Option 3)

14. Let A = {1, 2, 3, 4, 5, 6, 7}. Then the relation R = {(x,y) ∈A × A : x + y =7} is
(1) Symmetric but neither reflexive nor transitive
(2) Transitive but neither symmetric nor reflexive
(3) An equivalence relation
(4) Reflexive but neither symmetric nor transitive

Sol. **(1)**
R = {(1,6), (2,5), (3,4), (4,3),(5,2), (6,1)}

15. If the probability that the random variable X takes values x is given by P(X=x) = k (x+1)3^{-x}, x = 0, 1,2, 3, ..., where k is a constant, then P(X ≥ 2) is equal to

(1) $\frac{7}{27}$ (2) $\frac{11}{18}$ (3) $\frac{7}{18}$ (4) $\frac{20}{27}$

Sol. **(1)**

$$\sum_{x=0}^{\infty} P(X=x)=1$$

$k(1 + 2\cdot3^{-1} + 3\cdot3^{-2} + 4\cdot3^{-3} + \ldots\ \infty) = 1$

Let $s=1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots\infty$

$$\frac{s}{3}=\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^2}+\ldots\infty$$

$$\frac{2s}{3}=1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^2}+\ldots\infty$$

$$\frac{2s}{3}-\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$$

$$s=\frac{9}{4}$$

so $\boxed{k=\frac{4}{9}}$

$$P(X\geq 2)=1-P(x=0)-P(x=1)$$

$$=1-\frac{4}{9}\left(1+\frac{2}{3}\right)$$

$$=\frac{7}{27}\ \text{(Option 1)}$$

16. The integral $\int\left(\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x\right)$ $\log_2 x$ dx is equal to

(1) $\left(\frac{x}{2}\right)^x \log_2\left(\frac{2}{x}\right)+C$ (2) $\left(\frac{x}{2}\right)^x-\left(\frac{2}{x}\right)^x+C$

(3) $\left(\frac{x}{2}\right)^x \log_2\left(\frac{x}{2}\right)+C$ (4) $\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x+C$

Sol. **(2) Bonus**

$$\int\left(x^x2^{-x}+2^xx^{-x}\right)\log_2^x dx$$

$$\int\left(e^{x\ln x}\cdot e^{-x\ln 2}+e^{x\ln 2}\cdot e^{-x\ln x}\right)dx$$

$$\int\left(e^{x\ln x - x\ln 2} + e^{x\ln 2 - x\ln x}\right)\frac{\ln x}{\ln 2}dx$$

let $x\ln x - x\ln 2 = t$

$(\ln x + 1 - \ln 2)\,dx = dt$

17. The value of 36 $(4\cos^2 9^o - 1)(4\cos^2 27^o - 1)(4\cos^2 81^o - 1)(4\cos^2 243^o - 1)$ is

(1) 27 (2) 54 (3) 18 (4) 36

Sol. **(4)**

$4\cos^2\theta - 1 = 4(1-\sin^2\theta) - 1 = 3 - 4\sin^2\theta = \frac{\sin 3\theta}{\sin\theta}$

so given expression can be written as

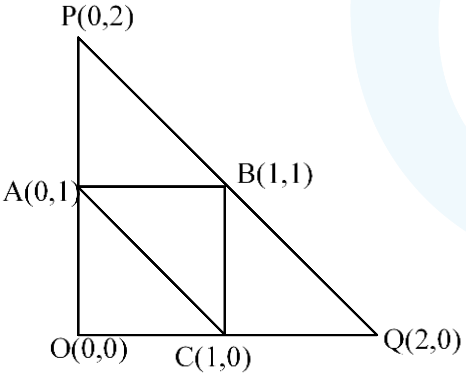
$$36\times\frac{\sin 27^o}{\sin 9^o}\times\frac{\sin 81^o}{\sin 27^o}\times\frac{\sin 243^o}{\sin 81^o}\times\frac{\sin 729^o}{\sin 243^o}$$

$$36\times\frac{\sin 729^o}{\sin 9^o} = 36$$

18. Let A (0, 1), B(1,1) and C (1,0) be the mid-points of the sides of a triangle with incentre at the point D. If the focus of the parabola $y^2 = 4ax$ passing through D is $\left(\alpha+\beta\sqrt{3},0\right)$, where α and β are rational numbers, then $\frac{\alpha}{\beta^2}$ is equal to

(1) 6 (2) 8 (3) $\frac{9}{2}$ (4) 12

Ans. **(2)**



$a = OP = 2$ $b = OQ = 2$ $c = PQ = 2\sqrt{2}$

(2,0) (0,2) (0,0)

$$D\left(\frac{4}{2+2+2\sqrt{2}},\frac{4}{2+2+2\sqrt{2}}\right)\equiv D\left(\frac{2}{2+\sqrt{2}},\frac{2}{2+\sqrt{2}}\right)$$

$$y^2 = 4ax \Rightarrow \left(\frac{2}{2+\sqrt{2}}\right)^2 = 4a\cdot\left(\frac{2}{2+\sqrt{2}}\right)$$

$$\therefore 4a = \frac{2}{2+\sqrt{2}}\ \therefore a = \frac{1}{2}\cdot\frac{2-\sqrt{2}}{4-2} = \frac{1}{4}\left(2-\sqrt{2}\right)$$

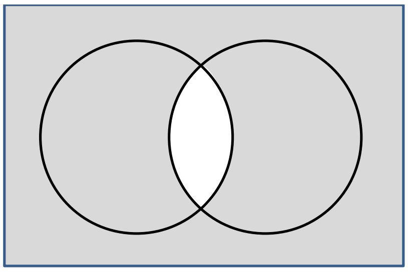
$$\therefore \alpha = \frac{2}{4} = \frac{1}{2} \qquad \beta = \frac{-1}{4}$$

$$\therefore \frac{\alpha}{\beta^2} = 8 \text{ Ans.}$$

19. The negation of $(p \wedge (\sim q)) \vee (\sim p)$ is equivalent to

(1) $p \wedge (\sim q)$ (2) $p \wedge (q \wedge (\sim p))$ (3) $p \vee (q \vee (\sim p))$ (4) $p \wedge q$

Sol. **4**



20. Let the mean and variance of 12 observations be $\frac{9}{2}$ and 4 respectively. Later on, it was observed that two observations were considered as 9 and 10 instead of 7 and 14 respectively. If the correct variance is $\frac{m}{n}$, where m and n are coprime, then m + n is equal to

(1) 316 (2) 317 (3) 315 (4) 314

Sol. **2**

$$\frac{\Sigma x}{12} = \frac{9}{2}$$

$$\sum x = 54$$

$$\frac{\Sigma x^2}{12} - \left(\frac{9}{2}\right)^2 = 4$$

$$\sum x^2 = 291$$

$$\sum x_{new} = 54 - (9 + 10) + 7 + 14 = 56$$

$$\sum x^2_{new} = 291 - (81 + 100) + 49 + 196 = 355$$

$$\sigma^2_{new} = \frac{355}{12} - \left(\frac{56}{12}\right)^2$$

$$\sigma^2_{new} = \frac{281}{36} = \frac{m}{n}$$

$\boxed{m + n = 317}$ Option (2)

SECTION - B

21. Let R = {a, b, c, d, e} and S = {1, 2, 3, 4}. Total number of onto functions f : R → S such that f(a) ≠ 1, is equal to ________.

Sol. **180**

Total onto function

$$\frac{\lfloor 5}{\lfloor 3 \, \lfloor 2} \times \lfloor 4 = 240$$

Now when f(a) = 1

$$\lfloor 4 + \frac{\lfloor 4}{\lfloor 2 \, \lfloor 2} \times \lfloor 3 = 24 + 36 = 60$$

so required $f^n = 240 - 60 = 180$

22. Let m and n be the numbers of real roots of the quadratic equations $x^2 - 12x + [x] + 31 = 0$ and $x^2 - 5|x + 2| - 4 = 0$ respectively, where [x] denotes the greatest integer $\leq x$. Then $m^2 + mn + n^2$ is equal to ________.

Sol. **9**

$x^2 - 12x + [x] + 31 = 0$

$\{x\} = x^2 - 11\ x + 31$

$0 \leq x^2 - 11x + 31 < 1$

$x^2 - 11x + 30 < 0$

$x \in (5, 6)$

so $[x] = 5$

$x^2 - 12x + 5 + 31 = 0$

$x^2 - 12x + 36 = 0$

$\boxed{x = 6}$ but $x \in (5, 6)$

so $x \in \phi$

$\boxed{m = 0}$

Now $x^2 - 5\,|x+2| - 4 = 0$

$x \geq -2$

$x^2 - 5x - 14 = 0$

$(x-7)(x+2) = 0$

$x = 7, -2$

$x < -2$

$x^2 + 5x + 6 = 0$

$(x+3)(x+2) = 0$

$x = -3, -2$

$x = \{7, -2, -3\}$

$n = 3$

$m^2 + mn + n^2 = n^2 = 9$

23. Let P_1 be the plane $3x - y - 7z = 11$ and P_2 be the plane passing through the points $(2, -1, 0)$, $(2, 0, -1)$, and $(5, 1, 1)$. If the foot of the perpendicular drawn from the point $(7, 4, -1)$ on the line of intersection of the planes P_1 and P_2 is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to _____.

Sol. **11**

P_2 is given by

$$\begin{vmatrix} x-5 & y-1 & z-1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 0$$

$\boxed{x - y - z = 3}$

DR of line intersection of P_1 & P_2

$$\begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 3 & -1 & -7 \end{vmatrix}$$

$+6\hat{i} + 4\hat{j} + 2\hat{k}$

Let $z = 0$, $x - y = 3$

$3x - y = 11$

$2x = 8$

$x = 4$

$y = 1$

So Line is

$$\frac{x-4}{6}=\frac{y-1}{4}=\frac{z-0}{2}=r$$

$(\alpha,\beta,\gamma) = (6r + 4, 4r + 1, 2r)$

$$6(\alpha-7)+4(\beta-4)+2(\gamma+1)=0$$

$$6\alpha-42+4\beta-16+2\gamma+2=0$$

$$36r+24+16r+4+4r-56=0$$

$$56r=28$$

$$r=\frac{1}{2}$$

$\alpha + \beta + \gamma = 12r + 5$

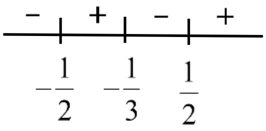
$= 6 + 5 = 11$

24. If domain of the function $\log_e\left(\frac{6x^2+5x+1}{2x-1}\right)+\cos^{-1}\left(\frac{2x^2-3x+4}{3x-5}\right)$ is $(\alpha,\beta) \cup (\gamma,\delta]$, then, $18(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ is equal to

Sol. **20**

$$\frac{6x^2+5x+1}{2x-1}>0$$

$$\frac{(3x+1)(2x+1)}{2x-1}>0$$

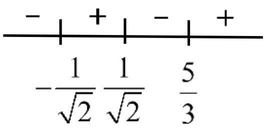


$$x\in\left(\frac{-1}{2},\frac{-1}{3}\right)\cup\left(\frac{1}{2},\infty\right) \qquad ....(A)$$

and

$$\underbrace{-1\le\frac{2x^2-3x+4}{3x-5}}\ \underbrace{\le 1}$$

$$\frac{2x^2-1}{3x-5}\ge 0 \qquad \text{and} \qquad \frac{2x^2-6x+9}{3x-5}\le 0$$



and $3x - 5 < 0$

$$x\in\left[\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]\cup\left(\frac{5}{3},\infty\right) \qquad ...(B)$$

$$x<\frac{5}{3} \qquad ...(C)$$

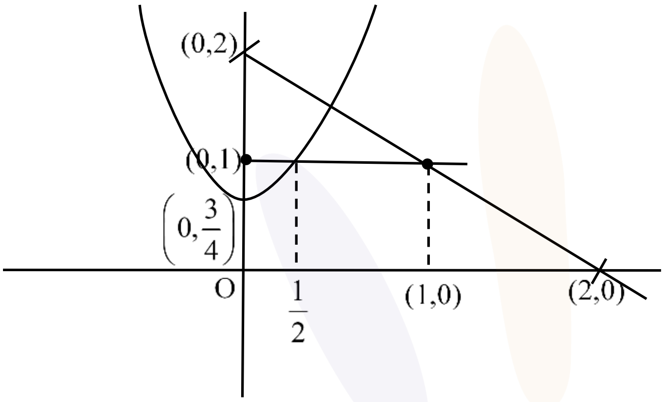
$$A\cap B\cap C\equiv\left(\frac{-1}{2},\frac{-1}{3}\right)\cup\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right]$$

So $18\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)=18\left(\frac{1}{4}+\frac{1}{9}+\frac{1}{4}+\frac{1}{2}\right)$

$= 18 + 2 = 20$

25. Let the area enclosed by the lines x + y = 2, y = 0 x = 0 and the curve $f(x)=\min\left\{x^2+\frac{3}{4},1+[x]\right\}$ where [x] denotes the greatest integer ≤ x, be A. Then the value of 12A is______.

Sol. **17**



$$\int_0^{\frac{1}{2}}\left(x^2+\frac{3}{4}\right)dx+\frac{1}{2}\times\left(\frac{3}{2}+\frac{1}{2}\right)\times 1$$

$$=\left[\frac{x^3}{3}+\frac{3x}{4}\right]_0^{\frac{1}{2}}+1$$

$$A=\frac{1}{24}+\frac{3}{8}+1$$

$$12A=\frac{1}{2}+\frac{36}{8}+12$$

$$=\frac{1}{2}+\frac{9}{2}+12$$

= 5 + 12

= 17

26. Let 0 < z < y < x be three real numbers such that $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in an arithmetic progression and x, $\sqrt{2}y,z$ are in a geometric progression. If xy + yz + zx = $\frac{3}{\sqrt{2}}$xyz , then $3(x+y+z)^2$ is equal to______.

Sol. **150**

$$\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$$

$$2y^2=xz$$

$$\frac{2}{y}=\frac{x+z}{xz}=\frac{x+z}{2y^2}$$

x + z = 4y

xy + yz + zx = $\frac{3}{\sqrt{2}}$xyz

y(x+z) + zx = $\frac{3}{\sqrt{2}}xz\cdot y$

$$4y^2 + 2y^2 = \frac{3}{\sqrt{2}} y \cdot 2y^2$$

$$6y^2 = 3\sqrt{2}y^3$$

$$y = \sqrt{2}$$

$x + y + z = 5y = 5\sqrt{2}$

$3(x + y + z)^2 = 3 \times 50 = 150$

27. Let the solution curve x = x (y), $0 < y < \frac{\pi}{2}$, of the differential equation $(\log_e (\cos y))^2 \cos y \, dx - (1 + 3x \log_e (\cos y)) \sin y \, dy = 0$ satisfy $x\left(\frac{\pi}{3}\right) = \frac{1}{2\log_e 2}$. If $x\left(\frac{\pi}{6}\right) = \frac{1}{\log_e m - \log_e n}$, where m and n are co-prime, then mn is equal to

Sol. **12**

Cos $y\ln^2 \cos y dx = (1 + 3x \ln \cos y)\sin y \, dy$

$$\frac{dx}{dy} = \tan y\left(\frac{3x}{\ln\cos y} + \frac{1}{\ln^2 \cos y}\right)$$

$$\frac{dx}{dy} - \left(\frac{3\tan y}{\ln\cos y}\right)x = \frac{\tan y}{\ln^2 \cos y}$$

If $= e^{-\frac{3}{\ln\cos y}\frac{\sin y}{\cos y}dy}$

$\ln \cos y = t$

$$\frac{1}{\cos y} - \sin y dy = dt$$

If $= e^{\frac{3}{t}dt} = e^{3\ln t} = t^3 = \ln^3 \cos y$

solution is $x \cdot \ln^3 \cos y = \{\frac{\sin y}{\cos y} \cdot \ln\cos y dy + C$

$$x\ln^3 \cos y = \frac{-\ln^2 \cos y}{2} + C$$

$x\left(\frac{\pi}{3}\right) = \frac{1}{2\ln 2}$ so $\frac{1}{2\ln 2} \times \ln^3\left(\frac{1}{2}\right) = -\frac{\ln^3\left(\frac{1}{2}\right)}{2} + C$

C = 0

$y = \frac{\pi}{6}$ $\qquad x\ln^3 \frac{\sqrt{3}}{2} = -\frac{1}{2}\ln^2 \frac{\sqrt{3}}{2} + 0$

$$x = -\frac{1}{2\ln\left(\frac{\sqrt{3}}{2}\right)}$$

$$x = \frac{1}{\ln\frac{4}{3}} = \frac{1}{\ln 4 - \ln 3}$$

mn = 12

28. Let [t] denote the greatest integer function. If $\int_{0}^{2.4}[x^2]dx=\alpha+\beta\sqrt{2}+\gamma\sqrt{3}+\delta\sqrt{5}$, then α + β + γ + δ is equal to _______.

Sol. **6**

$$\int_{0}^{1}0dx+\int_{1}^{\sqrt{2}}1dx+\int_{\sqrt{2}}^{\sqrt{3}}2dx+\int_{\sqrt{3}}^{2}3dx+\int_{2}^{\sqrt{5}}4dx+\int_{\sqrt{5}}^{2\cdot4}5dx$$

$$\sqrt{2}-1+2\left(\sqrt{3}-\sqrt{2}\right)+3\left(2-\sqrt{3}\right)+4\left(\sqrt{5}-2\right)+5\left(\left(2\cdot4\right)-\sqrt{5}\right)$$

$$=9-\sqrt{2}-\sqrt{3}-\sqrt{5}$$

$$\alpha+\beta+\gamma+\delta=9-1-1-1=6$$

29. The ordinates of the points P and Q on the parabola with focus (3.0) and directrix x=–3 are in the ratio 3 : 1 :. If R (α,β) is the point of intersection of the tangents to the parabola at P and Q, then $\frac{\beta^2}{\alpha}$ is equal to ________.

Sol. **16**

Parabola is $y^2 = 12x$

Let $Q(3t^2, 6t)$

so $P(27t^2, 18t)$

$$R(\alpha,\beta)=\left(at_1t_2,a\left(t_1+t_2\right)\right)$$

$$= (3t\cdot3t, 3(t + 3t))$$

$$R(\alpha,\beta) = (9t^2, 12t)$$

$$\frac{\beta^2}{\alpha}=\frac{(12t)^2}{9t^2}=\frac{144}{9}=16$$

30. Let k and m be positive real numbers such that the function $f\left(x\right)=\begin{cases}3x^2+k\sqrt{x+1}, 0<x<1\\ mx^2+k^2, \quad x\geq1\end{cases}$ is differentiable for all x > 0. Then $\frac{8f'(8)}{f'\left(\frac{1}{8}\right)}$ is equal to_______.

Sol. **309**

function is differentiable ∀x < 0

so $f(1^-) = f(1)$

$$3+\sqrt{2}k=m+k^2 \qquad ...(1)$$

and $f_+^1\left(1^-\right)=f_-^1\left(1^+\right)$

$$2mx\Big|_{x=1}=6x+\frac{k}{2\sqrt{x+1}}\Big|_{x=1}$$

$$2m=6+\frac{k}{2\sqrt{2}}$$

$$m=3+\frac{k}{4\sqrt{2}} \qquad ...(2)$$

$$k^2+3+\frac{k}{4\sqrt{2}}=3+\sqrt{2}k$$

$\boxed{k = \frac{7}{4\sqrt{2}}}, 0$

$$m = 3 + \frac{7}{32}$$

$$\boxed{m = \frac{103}{32}}$$

So $$\frac{8f'(8)}{f'\left(\frac{1}{8}\right)} = 8 \times \frac{2mx\big|_{x=8}}{6x + \frac{k}{2\sqrt{x+1}}\Big|_{x=\frac{1}{8}}}$$

$$= \frac{8 \times 2 \times 8 \times \frac{103}{32}}{\frac{16}{12}}$$

$$= 103 \times 3 = 309$$