FINAL JEE–MAIN EXAMINATION – APRIL, 2023
Held On Saturday 08th April, 2023
TIME : 09:00 AM to 12:00 PM

SECTION – A

1. The area of the region $\{(x, y): x^2 \le y \le 8 - x^2, y \le 7\}$ is.

(1) 24 (2) 21 (3) 20 (4) 18

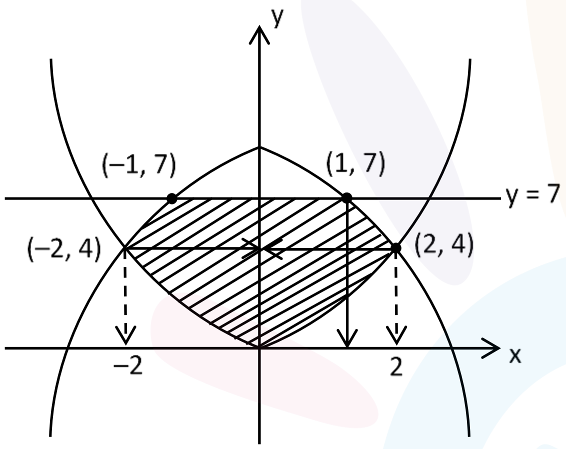
Sol. **(3)**

$y \ge x^2 \quad y \le 8 - x^2 \qquad y \le 7$

$x^2 = 8 - x^2$

$x^2 = 4$

$x = \pm 2$



$$2\left(1.7 + \int_1^2 (8 - 2x^2)dx\right) - 2\int_0^1 (x^2)dx$$

$$= 2\left[7 + \left(8x - \frac{2x^3}{3}\right)_1^2\right] - 2\left(\frac{x^3}{3}\right)_0^1$$

$$= 2\left[7 + \left(16 - \frac{16}{3}\right) - \left(8 - \frac{2}{3}\right)\right] - 2\left(\frac{1}{3}\right)$$

$$= 2\left[7 + \frac{32}{3} - \frac{22}{3}\right] = 2\left[7 + \frac{10}{3}\right]\frac{-2}{3}$$

$$= \frac{60}{3} = 20$$

2. Let $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$. If $P^T Q^{2007} P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then 2a+ b -3c – 4d equal to

(1) 2004 (2) 2007 (3) 2005 (4) 2006

Sol. **(3)**

$Q = PAP^T$

$P^T. Q^{2007}. P = P^T. Q.Q \ldots Q.P$

$= P^T(PAP^T) (P.AP^T) \ldots (PAP^T)P.$

$\Rightarrow (P^TP)A(P^TP)A \ldots A(P^TP)$

$$P^T.P = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore\ P^T.\ Q^{2007}.\ P = A^{2007}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^{2007} = \begin{bmatrix} 1 & 2007 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a = 1,\ b = 2007,\ c = 0,\ d = 1$

$2a + b - 3c - 4d = 2 + 2007 - 4 = 2005$

3. Negation of $(p \to q) \to (q \to p)$ is

(1) $(-q) \wedge p$ (2) $p \vee (\sim q)$ (3) $(\sim p) \vee q$ (4) $q \wedge (\sim p)$

Sol. **(4)**

$(p \to q) \to (q \to p)$

$\sim[\sim p \to q \wedge q \to p]$

$\Rightarrow\ p \to q \wedge \sim\ q \to p$

$\Rightarrow\ \sim p \vee q\ \wedge\ q \wedge \sim p$

$\Rightarrow q \wedge \sim p$.

4. Let $C(\alpha, \beta)$ be the circumcenter of the triangle formed by the lines

$4x + 3y = 69$,

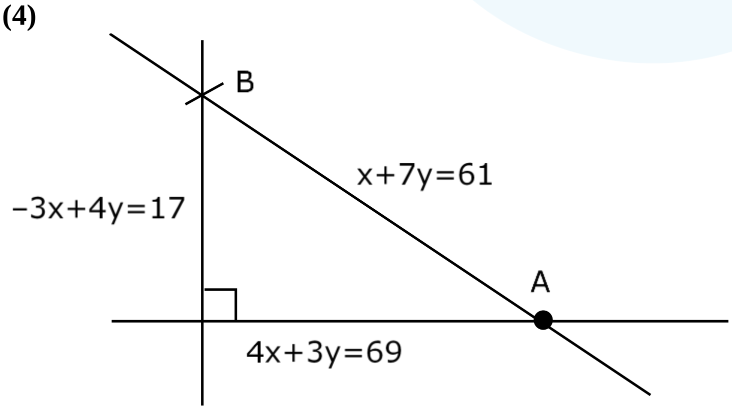
$4y - 3x = 17$ and

$x + 7y = 61$.

Then $(\alpha - \beta)^2 + \alpha + \beta$ is equal to

(1) 18 (2) 15 (3) 16 (4) 17

Sol. **(4)**



$4x + 28y = 244$

$4x + 3y = 69$

$- \quad - \quad -$

$25y = 175$

$y = 7,\ x = 12$

$A(12, 7)$

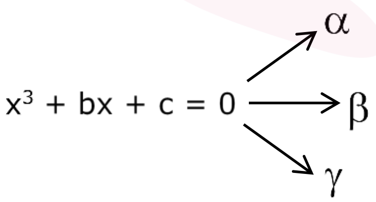
$-3x + 4y = 17$
$3x + 21y = 183$

$25y = 200$
$y = 8, x = 5$
B(5, 8)
$\therefore$ Circumcenter
$\alpha = \frac{17}{2} \beta = \frac{15}{2}$
$\left(\frac{17}{2}, \frac{15}{2}\right)$
$(\alpha - \beta)^2 + \alpha + \beta$
$1+16 = 17$

5. Let α, β, γ, be the three roots of the equation $x^3 + bx + c = 0$. If $\beta\gamma = 1 = -\alpha$, then $b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3$ is equal to

(1) $\frac{155}{8}$ (2) 21 (3) 19 (4) $\frac{169}{8}$

Sol. **(3)**



$\beta\gamma = 1$
$\alpha = -1$
Put $\alpha = -1$
$-1 - b + c = 0$
$c - b = 1$

also
$\alpha.\beta.\gamma = -c$
$-1 = -c \Rightarrow c = 1$
$\therefore b = 0$
$x^3 + 1 = 0$
$\alpha = -1, \beta = -w, \gamma = -w^2$
$\therefore b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3$
$0 + 2 + 3 + 6 + 8 = 19$

6. Let the number of elements in sets A and B be five and two respectively. Then the number of subsets of A × B each having at least 3 and at most 6 elements is:

(1) 752 (2) 772 (3) 782 (4) 792

Sol. **(4)**

$n(A\times B)=10$

$^{10}C_3+{}^{10}C_4+{}^{10}C_5+{}^{10}C_6=792$

7. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1 : 5 : 20, then the coefficient of the fourth term is

(1) 5481 (2) 3654 (3) 2436 (4) 1817

Sol. **(2)**

$\frac{^nC_r}{^nC_{r-1}}=5$ $\frac{^nC_{r+1}}{^nC_r}=4$

$\frac{n-r+1}{r}=5$ $n=5r+4\ \ldots(2)$

$n = 6r - 1\ \ldots(1)$

$\therefore n=29, r=5$

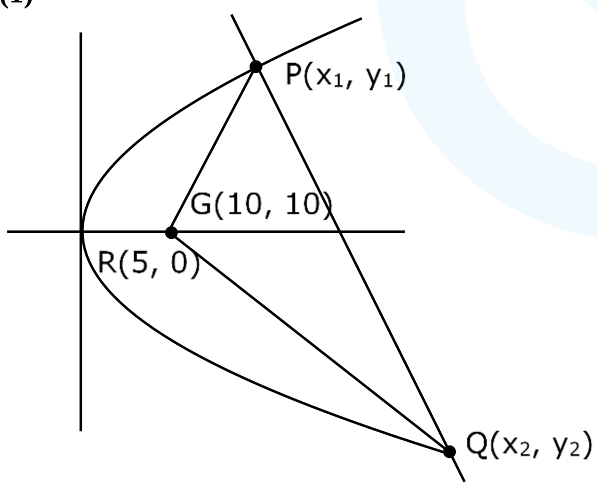
Coeff of 4^{th} term = $^{29}C_3$

= 3654

8. Let R be the focus of the parabola $y^2 = 20x$ and the line $y = mx + c$ intersect the parabola at two points P and Q. Let the point G(10, 10) be the centroid of the triangle PQR. If $c - m = 6$, then $(PQ)^2$ is

(1) 325 (2) 346 (3) 296 (4) 317

Sol. **(1)**



$y^2 = 20x$, $y = mx + c$

$y^2 = 20\left(\frac{y-c}{m}\right)$

$y^2-\frac{20y}{m}+\frac{20c}{m}=0$ $\frac{y_1+y_2+y_3}{3}=10$

$\frac{20}{m}=30$

$m=2/3$

and $c - m = 6$

$c = \frac{2}{3} + 6 \Rightarrow \frac{20}{3} = c$

$y^2 - 30y + \frac{20 \times 20/3}{2/3} = 0 \Rightarrow \quad y^2 - 30y + 200 = 0$

$y = 10, y = 20$

$y = 20, x = 20 \quad P(5, 10); (20, 20)Q$

$\frac{20+5+x}{3} = 10 \Rightarrow x = 5 \quad PQ^2 = 15^2 + 10^2 = 225 + 100 = 325$

9. Let $S_K \frac{1+2+...+K}{K}$ and $\sum_{j=1}^{n} S_j^2 = \frac{n}{A}(Bn^2 + Cn + D)$, where A, B, C, D $\in$N and A has least value. Then

(1) A + B is divisible by D (2) A + B = 5 (D –C)

(3) A + C + D is not divisible by B (4) A + B + D is divisible by 5

Sol. **(1)**

$S_k = \frac{k+1}{2}$

$S_k^2 = \frac{k^2 + 1 + 2k}{4}$

$\therefore \sum_{j-1}^{n} S_j^2 = \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + n + n(n+1)\right]$

$= \frac{n}{4}\left[\frac{(n+1)(2n+1)}{6} + 1 + n + 1\right]$

$= \frac{n}{4}\left[\frac{2n^2 + 3n + 1}{6} + n + 2\right]$

$= \frac{n}{4}\left[\frac{2n^2 + 9n + 13}{6}\right] = \frac{n}{24}\left[2n^2 + 9n + 13\right]$

A = 24, B = 2, C = 9, D = 13

10. The shortest distance between the lines $\frac{x-4}{4} = \frac{y+2}{5} = \frac{z+3}{3}$ and $\frac{x-1}{3} = \frac{y-3}{4} = \frac{z-4}{2}$ is

(1) $2\sqrt{6}$ (2) $3\sqrt{6}$ (3) $6\sqrt{3}$ (4) $6\sqrt{2}$

Sol. **(2)**

$S_d = \left|\frac{(\vec{a} - \vec{b}) \times (\vec{n}_1 \times \vec{n}_2)}{|\vec{n}_1 \times \vec{n}_2|}\right|$

$\vec{a} = (4, -2, -3)$

$\vec{b} = (1, 3, 4)$

$\vec{n}_1 = (4, 5, 3)$

$\vec{n}_2 = (3, 4, 2)$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 4 & 5 & 3 \\ 3 & 4 & 2 \end{vmatrix} = \hat{i}(-2) - \hat{j}(-1) + \hat{k}(1) = (-2, 1, 1)$

$S_d = \frac{(3, -5, -7) \cdot (-2, 1, 1)}{\sqrt{6}} = \left|\frac{-6 - 5 - 7}{\sqrt{6}}\right| = 3\sqrt{6}$

11. The number of arrangements of the letters of the word "INDEPENDENCE" in which all the vowels always occur together is.

(1) 16800 (2) 14800 (3) 18000 (4) 33600

Sol. **(1)**

IEEEE,

NNN, DD, P, C

$$\frac{8!}{3!2!}\times\frac{6!}{41}=16800$$

12. If the points with position vectors $\alpha\hat{i}+10\hat{j}+13\hat{k}$, $6\hat{i}+11\hat{j}+11\hat{k}$, $\frac{9}{2}\hat{i}+\beta\hat{j}-8\hat{k}$ are collinear, then $(19\alpha - 6\beta)^2$ is equal to

(1) 49 (2) 36 (3) 25 (4) 16

Sol. **(2)**

$$(\alpha,10,13);(6,11,11),\left(\frac{9}{2},\beta,-8\right)$$

$$\frac{\alpha-6}{3/2}=\frac{-1}{11-\beta}=\frac{2}{19}$$

$$\alpha-6=\frac{3}{19} \qquad -19=22-2\beta$$

$$\alpha=6+\frac{3}{19}=\frac{117}{19} \qquad 2\beta=41$$

$$\therefore(19\alpha-6\beta)^2=(117-123)^2=36$$

13. In a bolt factory, machines A, B and C manufacture respectively 20%, 30% and 50% of the total bolts. Of their output 3, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random form the product. If the bolt drawn is found the defective, then the probability that it is manufactured by the machine C is.

(1) $\frac{5}{14}$ (2) $\frac{3}{7}$ (3) $\frac{9}{28}$ (4) $\frac{2}{7}$

Sol. **(1)**

P(A) = $\frac{2}{10}$ P(B) = $\frac{3}{10}$ P(C) = $\frac{5}{10}$

P(Defective/A) = $\frac{3}{100}$, $P\left(\text{Defective}/B\right)=\frac{4}{100}$, $P\left(\text{Defective}/C\right)=\frac{2}{100}$

$$P(E)=\frac{5/10\times 2/100}{\frac{2}{10}\times\frac{3}{100}+\frac{3}{10}\times\frac{4}{100}+\frac{5}{10}\times\frac{2}{100}}=\frac{10}{6+12+10}$$

$$=\frac{10}{28}$$

$$=\frac{5}{14}$$

14. If for z = α + iβ, |z + 2| = z + 4(1 +i), then α + β and αβ are the roots of the equation

(1) $x^2 + 3x - 4 = 0$ (2) $x^2 + 7x + 12 = 0$

(3) $x^2 + x - 12 = 0$ (4) $x^2 + 2x - 3 = 0$

Sol. **(2)**

|z + 2| = |α + iβ + 2|

$= \alpha + i\beta + 4 + 4i$

$\sqrt{(\alpha+2)^2+\beta^2} = (\alpha+4)+i(\beta+4)$ $\quad\quad$ $\beta + 4 = 0$

$(\alpha+2)^2+16=(\alpha+4)^2$ $\quad\quad$ $\beta = -4$

$\alpha^2+4+4\alpha+16=\alpha^2+16+8\alpha$

$4 = 4\alpha$

$\alpha = 1$

$\alpha = 1, \beta = -4$

$\alpha + \beta = -3, \alpha\beta = -4$

Sum of roots = – 7

Product of roots = 12

$x^2 + 7x + 12 = 0$

15. $\lim_{x\to 0}\left(\left(\frac{(1-\cos^2(3x))}{\cos^3(4x)}\right)\left(\frac{\sin^3(4x)}{(\log_e(2x+1))^5}\right)\right)$ is equal to ______

(1) 24 (2) 9 (3) 18 (4) 15

Sol. **(3)**

$$\lim_{x\to 0}\left[\frac{1-\cos^2 3x}{9x^2}\right]\frac{9x^2}{\cos^3 4x}\cdot\frac{\left(\frac{\sin 4x}{4x}\right)^3\times 64x^3}{\left[\frac{\ln(1+2x)}{2x}\right]^5\times 32x^5}$$

$$\lim_{x\to 0}2\left(\frac{1}{2}\times\frac{9}{1}\times\frac{1\times 64}{1\times 32}\right)=18$$

16. The number of ways, in which 5 girls and 7 boys can be seated at a round table so that no two girls sit together, is

(1) $7(720)^2$ (2) 720 (3) $7(360)^2$ (4) $126(5!)^2$

Sol. **(4)**

$6!\times {}^7C_5 \times 5!$

$\Rightarrow 720\times 21\times 120$

$\Rightarrow 2\times 360\times 7\times 3\times 120$

$\Rightarrow 126\times(5!)^2$

17. Let $f(x)=\frac{\sin x+\cos x-\sqrt{2}}{\sin x-\cos x}, x\in[0.\pi]-\left\{\frac{\pi}{4}\right\}$. Then $f\left(\frac{7\pi}{12}\right)f''\left(\frac{7\pi}{12}\right)$ is equal to

(1) $\frac{-2}{3}$ (2) $\frac{2}{9}$ (3) $\frac{-1}{3\sqrt{3}}$ (4) $\frac{2}{3\sqrt{3}}$

Sol. **(2)**

$$f(x) = -\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

$$f'(x) = -\frac{1}{2}\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

$$f''(x) = -\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right)\cdot\tan\left(\frac{x}{2} \frac{-\pi}{8}\right)\cdot\frac{1}{2}$$

$$f\left(\frac{7\pi}{12}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

$$f''\left(\frac{7\pi}{12}\right) = -\frac{1}{2}\sec^2\frac{\pi}{6}.\tan\frac{\pi}{6} = \frac{-1}{2}\cdot\frac{4}{3}\times\frac{1}{\sqrt{3}} = \frac{-2}{3\sqrt{3}}$$

$$f\left(\frac{7\pi}{12}\right)\cdot f''\left(\frac{7\pi}{12}\right) = \frac{2}{9}$$

18. If the eqation of the plane containing the line x + 2y + 3z – 4 = 0 2x + y –z + 5 and perpendicular to the plane $\vec{r} = (\hat{i} - \hat{j}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$ is ax+by +cz = 4, then (a-b+c) is equal to

(1) 22 (2) 24 (3) 20 (4) 18

Sol. **(1)**

D.R's of line $\vec{n}_1 = -5\hat{i} + 7\hat{j} - 3\hat{k}$

D.R's of normal of second plane

$\vec{n}_2 = 5\hat{i} - 2\hat{j} - 3\hat{k}$

$\vec{n}_1 \times \vec{n}_2 = -27\hat{i} - 30\hat{j} - 25\hat{k}$

A point on the required plane is $\left(0, -\frac{11}{5}, \frac{14}{5}\right)$

The equation of required plane is

27x + 30y + 25z = 4

$\therefore$ a – b + c = 22

19. Let A = $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. If |adj(adj(adj2A))| =$(16)^n$, then n is equal to

(1) 8 (2) 9 (3) 12 (4) 10

Sol. **(4)**

|A| = 2[3] – 1[2] = 4

$\therefore \left|adj(adj(adj2A))\right|$

$= |2A|^{(n-1)^3} \Rightarrow |2A|^8 = 16^n$

$\Rightarrow (2^3|A|)^8 = 16^n$

$\Rightarrow (2^3 \times 2^2)^8 = 16^n$

$$= 2^{40} = 16^n$$
$$= 16^{10} = 16^n \Rightarrow n = 10$$

20. Let $I(x) = \int \frac{(x+1)}{x(1+x\,e^x)^2}\,dx,\ x>0$. If $\lim_{x\to\infty} I(x) = 0$, then I(1) is equal to

(1) $\frac{e+1}{e+2} - \log_e(e+1)$

(2) $\frac{e+2}{e+1} + \log_e(e+1)$

(3) $\frac{e+2}{e+1} - \log_e(e+1)$

(4) $\frac{e+1}{e+2} + \log_e(e+1)$

Sol.

$$I(x) = \int \frac{(x+1)}{x(1+xe^x)^2}dx$$

$$1 + xe^x = t$$
$$(xe^x + e^x)dx = dt$$
$$(x+1)dx = \frac{1}{e^x}dt$$

$$\therefore \int \frac{dt}{xe^x.t^2} \Rightarrow \int \frac{dt}{(t-1)t^2} \Rightarrow \int \frac{dt}{t(t-1)\cdot t} \Rightarrow \int \frac{t-(t-1)}{t(t-1)t}dt$$

$$\Rightarrow \int \frac{dt}{t(t-1)} - \int \frac{dt}{t^2} \Rightarrow \int \frac{t-(t-1)}{t(t-1)}dt + \frac{1}{t} + c$$

$$\Rightarrow \ln|t-1| - \ln|t| + \frac{1}{t} + c$$

$$\Rightarrow \ln|xe^x| - \ln|1+xe^x| + \frac{1}{1+xe^x} + c$$

$$I(x) = \ln\left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + c$$

$$\lim_{x\to\infty} I(x) = c = 0$$

$$\therefore I(1) = \ln\left|\frac{e}{1+e}\right| + \frac{1}{1+e}$$

$$= \ln e - \ln(1+e) + \frac{1}{1+e}$$

$$= \frac{e+2}{e+1} - \ln(1+e)$$

SECTION - B

21. Let A = {0,3 4, 6, 7, 8, 9, 10} and R be the relation defined on A such that R ={x,y) $\in$ A × A: x – y is odd positive integer or x – y = 2}. The minimum number of elements that must be aadded to the relation R, so that it is a symmetric relation, is equal to ______.

Sol. **(19)**

A = {0, 3, 4, 6, 7, 8, 9, 10} 3, 7, 9 $\rightarrow$ odd

$R=\{x-y=\text{odd}+\text{ve or } x-y=2\}$ 0, 4, 6, 8, 10 $\rightarrow$ even

${}^{3}C_1 \cdot {}^{5}C_1 = 15 +$ (6, 4), (8, 6), (10, 8), (9, 7)

Min^m ordered pairs to be added must be

: 15 + 4 = 19

22. Let (t) denote the greatest integer $\le$ t, If the constant term in the expansion of $\left(3x^2-\frac{1}{2x^5}\right)^7$ is α, then [α] is equal to ______.

Sol. **(1275)**

$$\left(3x^2-\frac{1}{2x^5}\right)^7$$

$$T_{r+1}={}^{7}C_r\left(3x^2\right)^{7-r}\left(-\frac{1}{2x^5}\right)^r$$

14 – 2r – 5r = 14 – 7r = 0

$\therefore r = 2$

$$\therefore T_3={}^{7}C_2.3^5\left(-\frac{1}{2}\right)^2=\frac{21\times 243}{4}=1275.75$$

$$\therefore[\alpha]=1275$$

23. Let λ_1, λ_2 be the values of λ for which the points $\left(\frac{5}{2},1,\lambda\right)$ and (–2, 0, 1) are at equal distance from the plane 2x + 3y – 6z + 7 = 0. If $\lambda_1 > \lambda_2$, then the distance of the point ($\lambda_1 - \lambda_2$, λ_2, λ_1) from the line $\frac{x-5}{1}=\frac{y-1}{2}=\frac{z+7}{2}$ is ______.

Sol. **9**

2x + 3y – 6z + 7 = 0 $\left(\frac{5}{2},1,\lambda\right),(-2,0,1)$

$$d_1=\left|\frac{5+3-6\lambda+7}{7}\right|=d_2=\left|\frac{-4-6+7}{7}\right|$$

$$\Rightarrow|15-6\lambda|=|3|$$

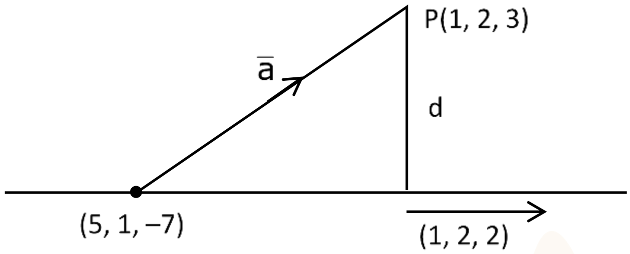
15 – 6λ = 3 or 15 – 6λ = – 3

6λ = 12 6λ = 18

λ = 2 λ = 3

λ_1 = 3, λ_2 = 2

$\therefore P(1,2,3)$ $\frac{x-5}{1}=\frac{y-1}{2}=\frac{z+7}{2}$



$$d = \left|\frac{(4,-1,-10)\times(1,2,2)}{3}\right| = \left|\frac{18\hat{i}-18\hat{j}+9\hat{k}}{3}\right| = 9$$

24. If the solution curve of the differential equation (y-2 $\log_e$ x)dx + (x $\log_e x^2$) dy = 0, x > 1 passes through the points $\left(e, \frac{4}{3}\right)$ and $\left(e^4, \alpha\right)$, then α is equal to ______.

Sol. **(3)**

$$(y - 2\ln x)dx + (2x\ln x)dy = 0$$

$$dy(2x\ln x) = [(2\ln x) - y]dx$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{y}{2x\ln x}$$

$$\frac{dy}{dx} + \frac{y}{2x\ln x} = \frac{1}{x}$$

$$\text{I.F} = e^{\int \frac{1}{2x\ln x}dx}$$

$$= e^{\frac{1}{2}\int\frac{dt}{t}} = e^{\frac{1}{2}\ln(\ln x)}$$

$$\Rightarrow \text{I.F} = (\ln x)^{1/2}$$

$$\therefore y\sqrt{\ln x} = \int \frac{\sqrt{\ln x}}{x}dx \qquad (\text{Let}, \ln x = u^2)$$

$$= 2\int u^2 du \qquad \frac{1}{x}dx = 2udu$$

$$y\sqrt{\ln x} = \frac{2}{3}(\ln x)^{3/2} + c \leftarrow \left(e, \frac{4}{3}\right)$$

$$\frac{4}{3} = \frac{2}{3} + c \Rightarrow c = \frac{2}{3}$$

$$y\sqrt{\ln x} = \frac{2}{3}(\ln x)^{3/2} + \frac{2}{3} \leftarrow (e^4, \alpha)$$

$$\alpha \cdot 2 = \frac{2}{3}\times 8 + \frac{2}{3}$$

$$\alpha = 3$$

25. Let $\vec{a} = 6\hat{i} + 9\hat{j} + 12\hat{k}, \vec{b} = \alpha\hat{i} + 11\hat{j} - 2\hat{k}$ and $\vec{c}$ be vectors such that $\vec{a} \times \vec{c} - = \vec{a} \times \vec{b}$. If $\vec{a}.\vec{c} = -12, \vec{c}.(\hat{i} - 2\hat{j} + \hat{k}) = 5$, then $\vec{c}.(\hat{i} + \hat{j} + \hat{k})$ is equal to ___.

Sol. **(11)**

$\vec{a} \times \vec{c} = \vec{a} \times 5$

$\Rightarrow \vec{a} \times \left(\vec{c} - \vec{b}\right) = 0$

$\vec{a} \,||^{rl} \left(\vec{c} - \vec{b}\right)$

$\therefore \vec{a} = \lambda\left(\vec{c} - \vec{b}\right)$

$(6,9,12) = \lambda[x - \alpha, y - 11, z + 2]$

$\frac{x-\alpha}{2} = \frac{y-11}{3} = \frac{z+2}{4}$

4y – 44 = 3z + 6

4y – 3z = 50

6x + 9y + 12z = – 12

2x + 3y + 4z = – 4 $\quad (\because x - 2y + z = 5)$

2x – 4y + 2z = 10

+ – –

7y + 2z = – 14 …(2)

8y – 6z = 100

21y + 6z = – 42

29y = 58

y = 2, z = – 14

$\therefore$ x – 4 – 14 = 5

x = 23

$\bar{c} = (23, 2, -14)$

$\bar{c} \cdot (1,1,1) = 23 + 2 - 14 = 11$

26. The largest natural number n such that 3^n divides 66! is ______.

Sol. **(31)**

$\left[\frac{66}{3}\right] + \left[\frac{66}{9}\right] + \left[\frac{66}{27}\right]$

22 + 7 + 2 = 31

27. If a_a is the greatest term in the sequence $a_n = \frac{n^3}{n^4 + 147}$, n = 1,2,3,..., then a is equal to ______.

Sol. **(0.158)**

$f(x) = \frac{x^3}{x^4 + 147}$

$f'(x) = \frac{\left(x^4 + 147\right)3x^2 - x^3\left(4x^3\right)}{\left(x^4 + 147\right)^2}$

$= \frac{3x^6 + 147 \times 3x^2 - 4x^6}{+ve} = x^2\left(44 - x^4\right)$

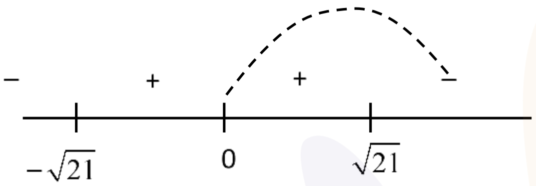
$f'(x) = 0$ at $x^6 = 147 \times 3x^2$

$x^2 = 0,\ x^4 = 147 \times 3$

$x = 0,\ x^2 = \pm\sqrt{147\times 3}$

$x^2 = \pm 21$

$x = \pm\sqrt{21}$



fmax at f(4) or f(5)

$f(4) = \frac{64}{403} \simeq 0.158 \qquad f(5) = \frac{125}{772} \simeq 0.161$

$\therefore\ a = 5$

28. Let the mean and variance of 8 numbers x, y, 10, 12, 6, 12, 4, 8 be 9 and 9.25 respectively. If $x > y$, then $3x - 2y$ is equal to ________.

Sol. **(25)**

$\frac{x+y+52}{8} = 9 \Rightarrow\ x + y = 20$

For variance

$x - 9,\ y - 9,\ 3,\ 3,\ 1,\ -5,\ -1,\ -3$

$\overline{x} = 0$

$\therefore\ \frac{(x-9)^2 + (y-9)^2 + 54}{8} - \overline{0}^2 = 9.25$

$(x-9)^2 + (11-x)^2 = 20$

$x = 7$ or $13\ \therefore\ y = 13, 7$

$3x - 2y = 3 \times 13 - 2 \times 7 = 25$

29. Consider a circle $C_1 : x^2 + y^2 - 4x - 2y = \alpha - 5$. Let its mirror image in the line $y = 2x + 1$ be another circle $C_2 : 5x^2 + 5y^2 - 10fx - 10gy + 36 = 0$. Let r be the radius of C_2. Then $\alpha + r$ is equal to ______.

Sol. **(2)**

$x^2 + y^2 - 4x - 2y + 5 - \alpha = 0,$

$C_1(2,1)\quad r_1 = \sqrt{\alpha}$

$2x - y + 1 = 0$

Image of (2, 1)

$\frac{x-2}{2} = \frac{y-1}{-1} = \frac{-2(4-1+1)}{5}$

$\frac{x-2}{2} = \frac{y-1}{-1} = \frac{-8}{5}$

$x = 2 - \frac{16}{5} = \frac{-6}{5},\ y = 1 + \frac{8}{5} = \frac{13}{5}$

$x^2 + y^2 - 2fx - 2gy + \frac{36}{5} = 0$

$C_2(f, g)$

$$r_2=\sqrt{f^2+g^2-\frac{36}{5}}$$

$$\alpha=f^2+g^2-\frac{36}{5}$$

$$\therefore f=-\frac{6}{5}, g=\frac{13}{5}$$

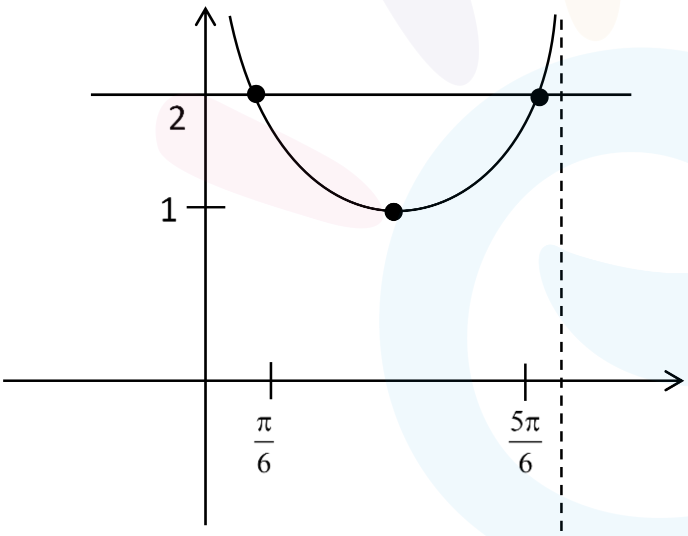
$$\alpha=\frac{36}{25}+\frac{169}{25}-\frac{36}{5}$$

$$=\frac{36+169-180}{25}\Rightarrow\alpha=1\Rightarrow r=1$$

$$\therefore \alpha+r=2$$

30. Let [t] denote the greatest integer $\leq$ t. The $\frac{2}{\pi}\int_{\pi/6}^{5\pi/6}(8[\operatorname{cosec} x]-5[\cot x])\,dx$ is equal to ______.

Sol. **(14)**



$$8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\operatorname{cosec}x]dx$$

$$8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}dx=\frac{16\pi/3}{16\pi/3}$$

$$I=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\cot x]dx$$

$$x\rightarrow\pi-x$$

$$I=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[-\cot x]dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left([\cot x] + [-\cot x]\right) dx$$

$$I = -\frac{1}{2}\int_{\pi/6}^{5\pi/6} dx \Rightarrow -\frac{1}{2}\left(\frac{4\pi}{6}\right)$$

$$= -\pi/3.$$

$$\therefore \frac{2}{\pi}\left[\frac{16\pi}{3} + \frac{5\pi}{3}\right] = \frac{2}{\pi}\left(\frac{21\pi}{3}\right)$$

$$= 14$$