FINAL JEE–MAIN EXAMINATION – AUGUST, 2021

(Held On Tuesday 31st August, 2021) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. If $\alpha + \beta + \gamma = 2\pi$, then the system of equations

$x + (\cos\gamma)y + (\cos\beta)z = 0$

$(\cos\gamma)x + y + (\cos\alpha)z = 0$

$(\cos\beta)x + (\cos\alpha)y + z = 0$

has :

(1) no solution

(2) infinitely many solution

(3) exactly two solutions

(4) a unique solution

Official Ans. by NTA (2)

Sol. $\alpha + \beta + \gamma = 2\pi$

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$$

$= 1 + 2\cos\alpha.\cos\beta.\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma$

$= \sin^2\gamma - \cos^2\alpha - \cos^2\beta + (\cos(\alpha+\beta) + \cos(\alpha-\beta))\cos\gamma$

$= \sin^2\gamma - \cos^2\alpha - \cos^2\beta + \cos^2\gamma + \cos(\alpha-\beta)\cos\gamma$

$= \sin^2\alpha - \cos^2\beta + \cos(\alpha-\beta)\cos(\alpha+\beta)$

$= \sin^2\alpha - \cos^2\beta + \cos^2\alpha - \sin^2\beta = 0$

2. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors mutually perpendicular to each other and have same magnitude. If a vector $\vec{r}$ satisfies.

$$\vec{a}\times\{(\vec{r}-\vec{b})\times\vec{a}\} + \vec{b}\times\{(\vec{r}-\vec{c})\times\vec{b}\} + \vec{c}\times\{(\vec{r}-\vec{a})\times\vec{c}\} = \vec{0},$$

then $\vec{r}$ is equal to :

(1) $\frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$

(2) $\frac{1}{3}(2\vec{a}+\vec{b}-\vec{c})$

(3) $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$

(4) $\frac{1}{2}(\vec{a}+\vec{b}+2\vec{c})$

Official Ans. by NTA (3)

Sol. Suppose $\vec{r} = x\vec{a} + y\vec{b} + 2\vec{c}$

and $|\vec{a}| = |\vec{b}| = |\vec{c}| = k$

$\vec{a}\times\{(\vec{r}-\vec{b})\times\vec{a}\} + \vec{b}\times\{(\vec{r}-\vec{c})\times\vec{b}\} + \vec{c}\times\{(\vec{r}-\vec{a})\times\vec{c}\} = \vec{0}$

$\Rightarrow k^2(\vec{r}-\vec{b}) - k^2x\vec{a} + k^2(\vec{r}-\vec{c}) - k^2y\vec{b} + k^2(\vec{r}-\vec{a}) - k^2z\vec{c} = \vec{0}$

$\Rightarrow 3\vec{r} - (\vec{a}+\vec{b}+\vec{c}) - \vec{r} = \vec{0}$

$\Rightarrow \vec{r} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{2}$

3. The domain of the function

$f(x) = \sin^{-1}\left(\dfrac{3x^2+x-1}{(x-1)^2}\right) + \cos^{-1}\left(\dfrac{x-1}{x+1}\right)$ is :

(1) $\left[0, \frac{1}{4}\right]$

(2) $[-2, 0] \cup \left[\frac{1}{4}, \frac{1}{2}\right]$

(3) $\left[\frac{1}{4}, \frac{1}{2}\right] \cup \{0\}$

(4) $\left[0, \frac{1}{2}\right]$

Official Ans. by NTA (3)

Sol. $f(x) = \sin^{-1}\left(\dfrac{3x^2+x-1}{(x-1)^2}\right) + \cos^{-1}\left(\dfrac{x-1}{x+1}\right)$

$-1 \le \dfrac{x-1}{x+1} \le 1 \Rightarrow 0 \le x < \infty$ …(1)

$-1 \le \dfrac{3x^2+x-1}{(x-1)^2} \le 1 \Rightarrow x \in \left[\dfrac{-1}{4}, \dfrac{1}{2}\right] \cup \{0\}$ …(2)

(1) & (2)

$\Rightarrow \text{Domain} = \left[\dfrac{1}{4}, \dfrac{1}{2}\right] \cup \{0\}$

4. Let S = {1, 2, 3, 4, 5, 6}. Then the probability that a randomly chosen onto function g from S to S satisfies g(3) = 2g(1) is :

(1) $\frac{1}{10}$

(2) $\frac{1}{15}$

(3) $\frac{1}{5}$

(4) $\frac{1}{30}$

Official Ans. by NTA (1)

Sol. g(3) = 2g(1) can be defined in 3 ways

number of onto functions in this condition = $3 \times 4!$

Total number of onto functions = 6!

Required probability = $\frac{3\times 4!}{6!} = \frac{1}{10}$

5. Let $f : \mathbf{N} \to \mathbf{N}$ be a function such that

$f(m + n) = f(m) + f(n)$ for every $m, n \in \mathbf{N}$. If $f(6) = 18$, then $f(2) \cdot f(3)$ is equal to :

(1) 6 (2) 54

(3) 18 (4) 36

Official Ans. by NTA (2)

Sol. $f(m + n) = f(m) + f(n)$

Put m = 1, n = 1

$f(2) = 2f(1)$

Put m = 2, n = 1

$f(3) = f(2) + f(1) = 3f(1)$

Put m = 3, n = 3

$f(6) = 2f(3) \Rightarrow f(3) = 9$

$\Rightarrow f(1) = 3, f(2) = 6$

$f(2).f(3) = 6 \times 9 = 54$

6. The distance of the point (–1, 2, –2) from the line of intersection of the planes 2x + 3y + 2z = 0 and x – 2y + z = 0 is :

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{5}{2}$

(3) $\frac{\sqrt{42}}{2}$ (4) $\frac{\sqrt{34}}{2}$

Official Ans. by NTA (4)

Sol. $P_1 : 2x + 3y + 2z = 0$

$\Rightarrow \vec{n}_1 = 2\hat{i} + 3\hat{j} + 2\hat{k}$

$P_2 : x - 2y + z = 0$

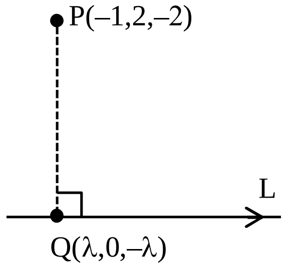
$\Rightarrow \vec{n}_2 = \hat{i} - 2\hat{j} + k$

Direction vector of line L which is line of intersection of P_1 & P_2

$\vec{r} = \vec{n}_1 \times \vec{n}_2 = 7\hat{i} - 7\hat{k}$

DR's of L are (1, 0, –1)

$\Rightarrow$ Equation of L : $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1} = \lambda$



DR's of $\overrightarrow{PQ} = (\lambda + 1, -2, 2 - \lambda)$

$\because \overrightarrow{PQ} \perp \vec{r}$

$\Rightarrow (\lambda + 1)(1) + (-2)(0) + (2 - \lambda)(-1) = 0$

$\Rightarrow \lambda = \frac{1}{2} \Rightarrow Q\left(\frac{1}{2}, 0, \frac{-1}{2}\right)$

$\Rightarrow PQ = \frac{\sqrt{34}}{2}$

7. Negation of the statement $(p \vee r) \Rightarrow (q \vee r)$ is :

(1) $p \wedge \sim q \wedge \sim r$ (2) $\sim p \wedge q \wedge \sim r$

(3) $\sim p \wedge q \wedge r$ (4) $p \wedge q \wedge r$

Official Ans. by NTA (1)

Sol. $\because \sim(A \Rightarrow B) = A \wedge \sim B$

$\therefore \sim((p \vee r) \Rightarrow (q \vee r))$

$= (p \vee r) \wedge (\sim q \wedge \sim r)$

$= ((p \vee r) \wedge (\sim r)) \wedge (\sim q)$

$= p \wedge (\sim r) \wedge (\sim q)$

8. If $\alpha = \lim\limits_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ and $\beta = \lim\limits_{x \to 0} (\cos x)^{\cot x}$ are the roots of the equation, $ax^2 + bx - 4 = 0$, then the ordered pair (a, b) is :

(1) (1, –3) (2) (–1, 3)

(3) (–1, –3) (4) (1, 3)

Official Ans. by NTA (4)

Sol. $\alpha = \lim\limits_{x \to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$; $\frac{0}{0}$ form

Using L Hopital rule

$\alpha = \lim\limits_{x \to \frac{\pi}{4}} \frac{3\tan^2 x \sec^2 x - \sec^2 x}{-\sin\left(x + \frac{\pi}{4}\right)}$

$\Rightarrow \alpha = -4$

$$\beta = \lim_{x\to 0}(\cos x)^{\cot x} = e^{\lim_{x\to 0}\frac{(\cos x - 1)}{\tan x}}$$

$$\beta = e^{\lim_{x\to 0}\frac{-(1-\cos x)}{x^2}\cdot\frac{x^2}{\left(\frac{\tan x}{x}\right)^x}}$$

$$\beta = e^{\lim_{x\to 0}\left(\frac{-1}{2}\right)\cdot\frac{x}{1}} = e^0 \Rightarrow \beta = 1$$

$\alpha = -4$; $\beta = 1$

If $ax^2 + bx - 4 = 0$ are the roots then

$16a - 4b - 4 = 0$ & $a + b - 4 = 0$

$\Rightarrow a = 1$ & $b = 3$

9. The locus of mid–points of the line segments joining (–3, –5) and the points on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ is :

(1) $9x^2+4y^2+18x+8y+145=0$

(2) $36x^2+16y^2+90x+56y+145=0$

(3) $36x^2+16y^2+108x+80y+145=0$

(4) $36x^2+16y^2+72x+32y+145=0$

Official Ans. by NTA (3)

Sol. General point on $\frac{x^2}{4}+\frac{y^2}{9}=1$ is $A(2\cos\theta, 3\sin\theta)$

given B(–3, –5)

midpoint $C\left(\frac{2\cos\theta-3}{2}, \frac{3\sin\theta-5}{2}\right)$

$$h=\frac{2\cos\theta-3}{2};\ k=\frac{3\sin\theta-5}{2}$$

$$\Rightarrow \left(\frac{2h+3}{2}\right)^2+\left(\frac{2k+5}{3}\right)^2=1$$

$\Rightarrow 36x^2 + 16y^2 + 108x + 80y + 145 = 0$

10. If $\frac{dy}{dx}=\frac{2^x y+2^y\cdot 2^x}{2^x+2^{x+y}\log_e 2}$, y(0) = 0, then for y = 1, the value of x lies in the interval:

(1) (1, 2) (2) $\left(\frac{1}{2},1\right]$

(3) (2, 3) (4) $\left(0,\frac{1}{2}\right]$

Official Ans. by NTA (1)

Sol. $$\frac{dy}{dx}=\frac{2^x(y+2^y)}{2^x(1+2^y\ell n2)}$$

$$\Rightarrow \int\frac{(1+2^y)\ell n2}{(y+2^y)}dy=\int dx$$

$\Rightarrow \ell n|y + 2^y| = x + c$

$x = 0;\ y = 0 \Rightarrow c = 0$

$\Rightarrow x = \ell n|y + 2^y|$

$\Rightarrow$ at $y = 1$, $x = \ell n3$

$\because 3 \in (e, e^2) \Rightarrow x \in (1, 2)$

11. An angle of intersection of the curves, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^2+y^2=ab$, $a > b$, is :

(1) $\tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$ (2) $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$

(3) $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$ (4) $\tan^{-1}\left(2\sqrt{ab}\right)$

Official Ans. by NTA (3)

Sol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $x^2 + y^2 = ab$

$$\frac{2x_1}{a^2}+\frac{2y_1y'}{b^2}=0$$

$$\Rightarrow y_1'=\frac{-x_1}{a^2}\frac{b^2}{y_1} \quad \ldots(1)$$

$\therefore 2x_1 + 2y_1y' = 0$

$$\Rightarrow y_2'=\frac{-x_1}{y_1} \quad \ldots(2)$$

Here (x_1y_1) is point of intersection of both curves

$$\therefore x_1^2=\frac{a^2b}{a+b},\ y_1^2=\frac{ab^2}{a+b}$$

$$\therefore \tan\theta=\left|\frac{y_1'-y_2'}{1+y_1'y_2'}\right|=\left|\frac{\frac{-x_1b^2}{a^2y_1}+\frac{x_1}{y_1}}{1+\frac{x_1^2b^2}{a^2y_1^2}}\right|$$

$$\tan\theta = \left|\frac{-b^2x_1y_1 + a^2x_1y_1}{a^2y_1^2 + b^2x_1^2}\right|$$

$$\tan\theta = \left|\frac{a-b}{\sqrt{ab}}\right|$$

12. If $y\frac{dy}{dx} = x\left[\frac{y^2}{x^2} + \frac{\phi\left(\frac{y^2}{x^2}\right)}{\phi'\left(\frac{y^2}{x^2}\right)}\right], x>0, \phi>0, \text{ and } y(1) = -1,$

then $\phi\left(\frac{y^2}{4}\right)$ is equal to :

(1) 4 ϕ (2) (2) 4 ϕ (1)

(3) 2 ϕ (1) (4) ϕ (1)

Official Ans. by NTA (2)

Sol. Let, y = tx

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$\therefore\ tx\left(t + x\frac{dt}{dx}\right) = x\left(t^2 + \frac{\varphi(t^2)}{\varphi'(t^2)}\right)$$

$$t^2 + xt\frac{dt}{dx} = t^2 + \frac{\varphi(t^2)}{\varphi'(t^2)}$$

$$\int\frac{t\varphi'(t^2)}{\varphi(t^2)}dt = \int\frac{dx}{x}$$

Let $\varphi(t^2) = p$

$\therefore \varphi'(t^2)2tdt = dp$

$$\Rightarrow \int\frac{dy}{2p} = \int\frac{dx}{x}$$

$$\frac{1}{2}\ell n\varphi(t^2) = \ell nx + \ell nc$$

$\varphi(t^2) = x^2k$

$$\varphi\left(\frac{y^2}{x^2}\right) = kx^2, \varphi(1) = k$$

$$\varphi\left(\frac{y^2}{4}\right) = 4\varphi(1)$$

13. The sum of the roots of the equation $x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0$, is :

(1) $\log_2 14$ (2) $\log_2 11$

(3) $\log_2 12$ (4) $\log_2 13$

Official Ans. by NTA (2)

Sol. $x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0$

$\log_2(2^{x+1}) - \log_2(3 + 2^x)^2 + \log_2(10 - 2^{-x}) = 0$

$$\log_2\left(\frac{2^{x+1}.(10 - 2^{-x})}{(3 + 2^x)^2}\right) = 0$$

$$\frac{2(10.2^x - 1)}{(3 + 2^x)^2} = 1$$

$\Rightarrow 20.2^x - 2 = 9 + 2^{2x} + 6.2^x$

$\therefore (2^x)^2 - 14(2^x) + 11 = 0$

Roots are 2^{x_1} & 2^{x_2}

$\therefore 2^{x_1}.2^{x_2} = 11$

$x_1 + x_2 = \log_2(11)$

14. If z is a complex number such that $\frac{z-i}{z-1}$ is purely imaginary, then the minimum value of |z – (3 + 3i)| is :

(1) $2\sqrt{2} - 1$ (2) $3\sqrt{2}$

(3) $6\sqrt{2}$ (4) $2\sqrt{2}$

Official Ans. by NTA (4)

Sol. $\frac{z-i}{z-1}$ is purely Imaginary number

Let z = x + iy

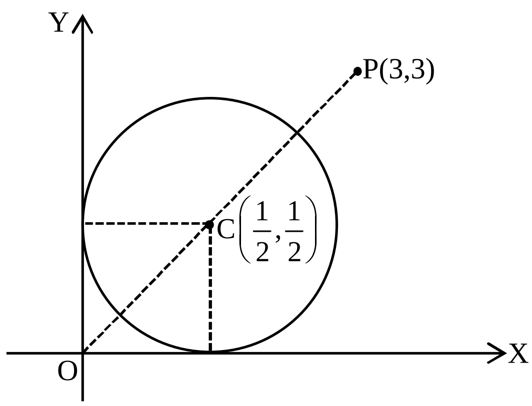
$$\therefore \frac{x + i(y-1)}{(x-1) + i(y)} \times \frac{(x-1) - iy}{(x-1) - iy}$$

$$\Rightarrow \frac{x(x-1) + y(y-1) + i(-y - x + 1)}{(x-1)^2 + y^2}$$ is purely

Imaginary number

$\Rightarrow x(x - 1) + y(y - 1) = 0$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$



$\therefore |z-(3+3i)|_{min} = |PC| - \frac{1}{\sqrt{2}}$

$= \frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2\sqrt{2}$

15. Let $a_1, a_2, a_3, \ldots$ be an A.P. If $\frac{a_1+a_2+\ldots+a_{10}}{a_1+a_2+\ldots+a_p} = \frac{100}{p^2}$, $p \neq 10$, then $\frac{a_{11}}{a_{10}}$ is equal to :

(1) $\frac{19}{21}$ (2) $\frac{100}{121}$

(3) $\frac{21}{19}$ (4) $\frac{121}{100}$

Official Ans. by NTA (3)

Sol. $\frac{\frac{10}{2}(2a_1+9d)}{\frac{p}{2}(2a_1+(p-1)d)} = \frac{100}{p^2}$

$(2a_1 + 9d)p = 10(2a_1 + (p-1)d)$

$9dp = 20a_1 - 2pa_1 + 10d(p-1)$

$9p = (20-2p)\frac{a_1}{d} + 10(p-1)$

$\frac{a_1}{d} = \frac{(10-p)}{2(10-p)} = \frac{1}{2}$

$\therefore \frac{a_{11}}{a_{10}} = \frac{a_1+10d}{a_1+9d} = \frac{\frac{1}{2}+10}{\frac{1}{2}+9} = \frac{21}{19}$

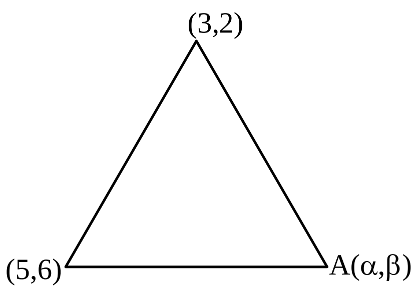
16. Let A be the set of all points (α, β) such that the area of triangle formed by the points (5, 6), (3, 2) and (α, β) is 12 square units. Then the least possible length of a line segment joining the origin to a point in A, is :

(1) $\frac{4}{\sqrt{5}}$ (2) $\frac{16}{\sqrt{5}}$

(3) $\frac{8}{\sqrt{5}}$ (4) $\frac{12}{\sqrt{5}}$

Official Ans. by NTA (3)

Sol.



$\left|\frac{1}{2}\begin{vmatrix} 5 & 6 & 1 \\ 3 & 2 & 1 \\ \alpha & \beta & 1 \end{vmatrix}\right| = 12$

$4\alpha - 2\beta = \pm 24 + 8$

$\Rightarrow 4\alpha - 2\beta = +24 + 8 \Rightarrow 2\alpha - \beta = 16$

$2x - y - 16 = 0$ ...(1)

$\Rightarrow 4\alpha - 2\beta = -24 + 8 \Rightarrow 2\alpha - \beta = -8$

$2x - y + 8 = 0$ ...(2)

perpendicular distance of (1) from (0, 0)

$\left|\frac{0-0-16}{\sqrt{5}}\right| = \frac{16}{\sqrt{5}}$

perpendicular distance of (2) from (0, 0) is

$\left|\frac{0-0+8}{\sqrt{5}}\right| = \frac{8}{\sqrt{5}}$

17. The number of solutions of the equation $32^{\tan^2 x} + 32^{\sec^2 x} = 81$, $0 \le x \le \frac{\pi}{4}$ is :

(1) 3 (2) 1

(3) 0 (4) 2

Official Ans. by NTA (2)

Sol. $(32)^{\tan^2 x} + (32)^{\sec^2 x} = 81$

$\Rightarrow (32)^{\tan^2 x} + (32)^{1+\tan^2 x} = 81$

$\Rightarrow (32)^{\tan^2 x} = \frac{81}{33}$

In interval $\left[0, \frac{\pi}{4}\right]$ only one solution

18. Let f be any continuous function on [0, 2] and twice differentiable on (0, 2). If f(0) = 0, f(1) = 1 and f(2) = 2, then

(1) $f''(x) = 0$ for all $x \in (0, 2)$

(2) $f''(x) = 0$ for some $x \in (0, 2)$

(3) $f'(x) = 0$ for some $x \in [0, 2]$

(4) $f''(x) > 0$ for all $x \in (0, 2)$

Official Ans. by NTA (2)

Sol. $f(0) = 0$ $f(1) = 1$ and $f(2) = 2$

Let $h(x) = f(x) - x$ has three roots

By Rolle's theorem $h'(x) = f'(x) - 1$ has at least two roots

$h''(x) = f''(x) = 0$ has at least one roots

19. If [x] is the greatest integer $\le$ x, then

$$\pi^2 \int_0^2 \left(\sin\frac{\pi x}{2}\right)(x-[x])^{[x]}\,dx \text{ is equal to :}$$

(1) $2(\pi - 1)$ (2) $4(\pi - 1)$

(3) $4(\pi + 1)$ (4) $2(\pi + 1)$

Official Ans. by NTA (2)

Sol. $$\pi^2\left[\int_0^1 \sin\frac{\pi x}{2}dx + \int_1^2 \sin\frac{\pi x}{2}(x-1)\,dx\right]$$

$$=\pi^2\left[-\frac{2}{\pi}\left(\cos\frac{\pi x}{2}\right)+\left((x-1)\left(-\frac{2}{\pi}\cos\frac{\pi x}{2}\right)\right)_1^2-\int_1^2 -\frac{2}{\pi}\cos\frac{\pi x}{2}dx\right]$$

$$=\pi^2\left[0+\frac{2}{\pi}+\frac{2}{\pi}+\frac{2}{\pi}.\frac{2}{\pi}\left(\sin\frac{\pi x}{2}\right)_1^2\right]$$

$= 4\pi - 4 = 4(\pi - 1)$

20. The mean and variance of 7 observations are 8 and 16 respectively. If two observations are 6 and 8, then the variance of the remaining 5 observations is :

(1) $\frac{92}{5}$ (2) $\frac{134}{5}$

(3) $\frac{536}{25}$ (4) $\frac{112}{5}$

Official Ans. by NTA (3)

Sol. Let 8, 16, x_1, x_2, x_3, x_4, x_5 be the observations.

Now $\frac{x_1 + x_2 + ... + x_5 + 14}{7} = 8$

$\Rightarrow \sum_{i=1}^{5} x_i = 42$ ...(1)

Also $\frac{x_1^2 + x_2^2 + ...x_5^2 + 8^2 + 6^2}{7} - 64 = 16$

$\Rightarrow \sum_{i=1}^{5} x_i^2 = 560 - 100 = 460$ ...(2)

So variance of $x_1, x_2, \ldots, x_5$

$$= \frac{460}{5} - \left(\frac{42}{5}\right)^2 = \frac{2300-1764}{25} = \frac{536}{25}$$

SECTION-B

1. If the coefficient of a^7b^8 in the expansion of $(a + 2b + 4ab)^{10}$ is $K.2^{16}$, then K is equal to ______.

Official Ans. by NTA (315)

Sol. $\frac{10!}{\alpha!\beta!\gamma!}a^{\alpha}(2b)^{\beta}.(4ab)^{\gamma}$

$\frac{10!}{\alpha!\beta!\gamma!}a^{\alpha+\gamma}.b^{\beta+\gamma}.2^{\beta}.4^{\gamma}$

$\alpha + \beta + \gamma = 10$ ...(1)

$\alpha + \gamma = 7$ ...(2)

$\beta + \gamma = 8$ ...(3)

(2) + (3) – (1) $\Rightarrow \gamma = 5$

$\alpha = 2$

$\beta = 3$

so coefficients $= \frac{10!}{2!3!5!}2^3.2^{10}$

$= \frac{10\times9\times8\times7\times6\times5}{2\times3\times2\times5!}\times 2^{13}$

$= 315 \times 2^{16} \Rightarrow k = 315$

2. Suppose the line $\frac{x-2}{\alpha} = \frac{y-2}{-5} = \frac{z+2}{2}$ lies on the plane $x + 3y - 2z + \beta = 0$. Then $(\alpha + \beta)$ is equal to ____.

Official Ans. by NTA (7)

Sol. Point (2, 2, –2) also lies on given plane

So $2 + 3 \times 2 - 2(-2) + \beta = 0$

$\Rightarrow 2 + 6 + 4 + \beta = 0 \Rightarrow \beta = -12$

Also $\alpha \times 1 - 5 \times 3 + 2 \times -2 = 0$

$\Rightarrow \alpha - 15 - 4 = 0 \Rightarrow \alpha = 19$

$\therefore \alpha + \beta = 19 - 12 = 7$

3. The number of 4-digit numbers which are neither multiple of 7 nor multiple of 3 is ____.

Official Ans. by NTA (5143)

Sol. A = 4 – digit numbers divisible by 3

A = 1002, 1005, …, 9999.

$9999 = 1002 + (n - 1)3$

$\Rightarrow (n-1)3 = 8997 \Rightarrow n = 3000$

B = 4 – digit numbers divisible by 7

B = 1001, 1008, …, 9996

$\Rightarrow 9996 = 1001 + (n-1)7$

$\Rightarrow n = 1286$

$A \cap B = 1008, 1029, \ldots, 9996$

$9996 = 1008 + (n-1)21$

$\Rightarrow n = 429$

So, no divisible by either 3 or 7

$= 3000 + 1286 - 429 = 3857$

total 4-digits numbers = 9000

required numbers = 9000 – 3857 = 5143

4. If $\int \frac{\sin x}{\sin^3 x + \cos^3 x} dx =$

$\alpha \log_e |1+\tan x| + \beta \log_e |1-\tan x + \tan^2 x| + \gamma \tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C,$

when C is constant of integration, then the value of $18(\alpha + \beta + \gamma^2)$ is ________.

Official Ans. by NTA (3)

Sol. $= \int \frac{\frac{\sin x}{\cos^3 x}}{1+\tan^3 x} dx = \int \frac{\tan x . \sec^2 x}{(\tan x + 1)(1+\tan^2 x - \tan x)} dx$

Let $\tan x = t \Rightarrow \sec^2 x . dx = dt$

$= \int \frac{t}{(t+1)(t^2-t+1)} dt$

$= \int \left(\frac{A}{t+1} + \frac{B(2t-1)}{t^2-t+1} + \frac{C}{t^2-t+1}\right) dx$

$\Rightarrow A(t^2 - t + 1) + B(2t - 1)(t^2 - t + 1) + C(t + 1) = t$

$\Rightarrow t^2(A + 2B) + t(-A + B + C) + A - B + C = 1$

$\therefore A + 2B = 0$ …(1)

$-A + B + C = 1$ …(2)

$A - B + C = 0$ …(3)

$\Rightarrow C = \frac{1}{2} \Rightarrow A - B = -\frac{1}{2}$ …(4)

$A + 2B = 0$

$A - B = -\frac{1}{2}$

$\Rightarrow 3B = \frac{1}{2} \Rightarrow B = \frac{1}{6}$

$A = -\frac{1}{3}$

$I = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2}\int \frac{dt}{t^2-t+1}$

$= -\frac{1}{3} \ell n|(1 + \tan x)| + \frac{1}{6} \ell n|\tan^2 x - \tan x + 1|$

$+ \frac{1}{2} . \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\left(\tan x - \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right)$

$= -\frac{1}{3} \ell n|(1 + \tan x)| + \frac{1}{6} \ell n|\tan^2 x - \tan x + 1|$

$+ \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C$

$\alpha = -\frac{1}{3}, \beta = \frac{1}{6}, \gamma = \frac{1}{\sqrt{3}}$

$18(\alpha + \beta + \gamma^2) = 18\left(-\frac{1}{3} + \frac{1}{6} + \frac{1}{3}\right) = 3$

5. A tangent line L is drawn at the point (2, –4) on the parabola $y^2 = 8x$. If the line L is also tangent to the circle $x^2 + y^2 = a$, then 'a' is equal to________.

Official Ans. by NTA (2)

Sol. tangent of $y^2 = 8x$ is $y = mx + \frac{2}{m}$

$P(2, -4) \Rightarrow -4 = 2m + \frac{2}{m}$

$\Rightarrow m + \frac{1}{m} = -2 \Rightarrow m = -1$

$\therefore$ tangent is $y = -x - 2$

$\Rightarrow x + y + 2 = 0$ …(1)

(1) is also tangent to $x^2 + y^2 = a$

So $\frac{2}{\sqrt{2}} = \sqrt{a} \Rightarrow \sqrt{a} = \sqrt{2}$

$\Rightarrow a = 2$

6. If $S = \frac{7}{5} + \frac{9}{5^2} + \frac{13}{5^3} + \frac{19}{5^4} + \ldots$, then 160 S is equal to _______.

Official Ans. by NTA (305)

Sol.

$$S = \frac{7}{5} + \frac{9}{5^2} + \frac{13}{5^3} + \frac{19}{5^4} + \ldots$$

$$\frac{1}{5}S = \qquad \frac{7}{5^2} + \frac{9}{5^3} + \frac{13}{5^4} + \ldots$$

On subtracting

$$\frac{4}{5}S = \frac{7}{5} + \frac{2}{5^2} + \frac{4}{5^3} + \frac{6}{5^4} + \ldots$$

$$S = \frac{7}{4} + \frac{1}{10}\left(1 + \frac{2}{5} + \frac{3}{5^2} + \ldots\right)$$

$$S = \frac{7}{4} + \frac{1}{10}\left(1 - \frac{1}{5}\right)^{-2}$$

$$= \frac{7}{4} + \frac{1}{10} \times \frac{25}{16} = \frac{61}{32}$$

$\Rightarrow 160S = 5 \times 61 = 305$

7. The number of elements in the set

$$\left\{A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \{-1, 0, 1\} \text{ and } (I - A)^3 = I - A^3\right\},$$

where I is 2×2 identity matrix, is :

Official Ans. by NTA (8)

Sol. $(I - A)^3 = I^3 - A^3 - 3A(I - A) = I - A^3$

$\Rightarrow 3A(I - A) = 0$ or $A^2 = A$

$$\Rightarrow \begin{bmatrix} a^2 & ab + bd \\ 0 & d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$\Rightarrow a^2 = a,\ b(a + d - 1) = 0,\ d^2 = d$

If $b \neq 0$, $a + d = 1 \Rightarrow$ 4 ways

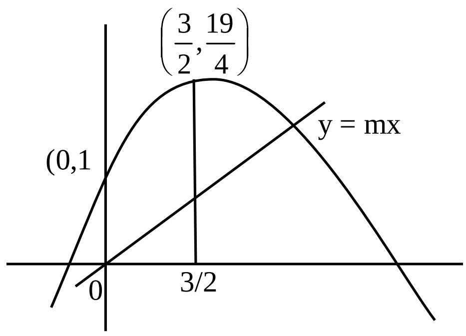
If $b = 0$, $a = 0, 1$ & $d = 0, 1 \Rightarrow$ 4 ways

$\Rightarrow$ Total 8 matrices

8. If the line $y = mx$ bisects the area enclosed by the lines $x = 0$, $y = 0$, $x = \frac{3}{2}$ and the curve $y = 1 + 4x - x^2$, then 12 m is equal to ________.

Official Ans. by NTA (26)

Sol.



$$\text{Total area} = \int_0^{3/2} (1 + 4x - x^2)\,dx$$

$$= x + 2x^2 - \frac{x^3}{3}\Bigg|_0^{3/2} = \frac{39}{8}$$

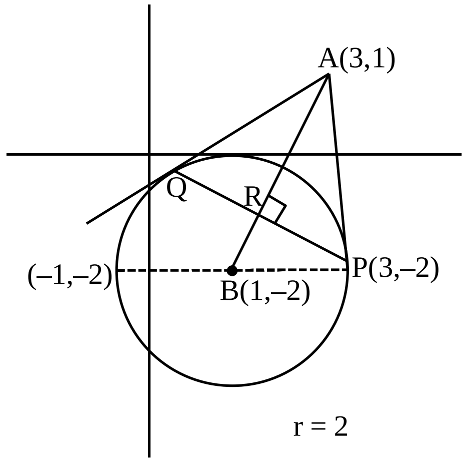
$$\& \ \frac{39}{16} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} m$$

$$\Rightarrow 3m = \frac{13}{2} \Rightarrow 12m = 26$$

9. Let B be the centre of the circle $x^2 + y^2 - 2x + 4y + 1 = 0$. Let the tangents at two points P and Q on the circle intersect at the point A(3, 1). Then $8.\left(\frac{\text{area } \Delta APQ}{\text{area } \Delta BPQ}\right)$ is equal to _____.

Official Ans. by NTA (18)

Sol.



$\tan\theta = \frac{3}{2}$

$$\frac{\text{Area } \Delta APQ}{\text{Area } \Delta BPQ} = \frac{AR}{RB} = \frac{3\sin\theta}{2\cos\theta} = \frac{9}{4}$$

$$8\left(\frac{\text{Area } \Delta APQ}{\text{Area } \Delta BPQ}\right) = 18$$

10. Let f(x) be a cubic polynomial with f(1) = –10, f(–1) = 6, and has a local minima at x = 1, and f'(x) has a local minima at x = –1. Then f(3) is equal to _____.

Official Ans. by NTA (22)

Sol. $F'(x) = a(x - 1)(x + 3)$

$F''(x) = 6a(x + 1)$

$F'(x) = 3a(x + 1)^2 + b$

$F'(1) = 0 \Rightarrow b = -12a$

$F(x) = a(x + 1)^3 - 12ax + c$

$= (x + 1)^3 - 12x - 6$

$F(3) = 64 - 36 - 6 = 22$