FINAL JEE–MAIN EXAMINATION – AUGUST, 2021

(Held On Friday 27th August, 2021) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. If $0 < x < 1$, then $\frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 + ...$, is equal to :

(1) $x\left(\frac{1+x}{1-x}\right) + \log_e(1-x)$

(2) $x\left(\frac{1-x}{1+x}\right) + \log_e(1-x)$

(3) $\frac{1-x}{1+x} + \log_e(1-x)$

(4) $\frac{1+x}{1-x} + \log_e(1-x)$

Official Ans. by NTA (1)

Sol. Let $t = \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 + ...\infty$

$$= \left(2 - \frac{1}{2}\right)x^2 + \left(2 - \frac{1}{3}\right)x^3 + \left(2 - \frac{1}{4}\right)x^4 + ...\infty$$

$$= 2\left(x^2 + x^3 + x^4 + ...\infty\right) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty\right)$$

$$= \frac{2x^2}{1-x} - \left(\ell n(1-x) - x\right)$$

$$\Rightarrow t = \frac{2x^2}{1-x} + x - \ell n(1-x)$$

$$\Rightarrow t = \frac{x(1+x)}{1-x} - \ell n(1-x)$$

2. If for $x, y \in \mathbf{R}$, $x > 0$,

$y = \log_{10}x + \log_{10}x^{1/3} + \log_{10}x^{1/9} + ...$ upto ∞ terms

and $\frac{2+4+6+....+2y}{3+6+9+....+3y} = \frac{4}{\log_{10} x}$, then the ordered pair (x, y) is equal to :

(1) $(10^6, 6)$ (2) $(10^4, 6)$

(3) $(10^2, 3)$ (4) $(10^6, 9)$

Official Ans. by NTA (4)

Sol. $\frac{2(1+2+3+...+y)}{3(1+2+3+...+y)} = \frac{4}{\log_{10} x}$

$\Rightarrow \log_{10}x = 6 \Rightarrow x = 10^6$

Now,

$$y = (\log_{10} x) + \left(\log_{10} x^{\frac{1}{3}}\right) + \left(\log_{10} x^{\frac{1}{9}}\right) + ..\infty$$

$$= \left(1 + \frac{1}{3} + \frac{1}{9} + ...\infty\right)\log_{10} x$$

$$= \left(\frac{1}{1-\frac{1}{3}}\right)\log_{10} x = 9$$

So, $(x,y) = (10^6, 9)$

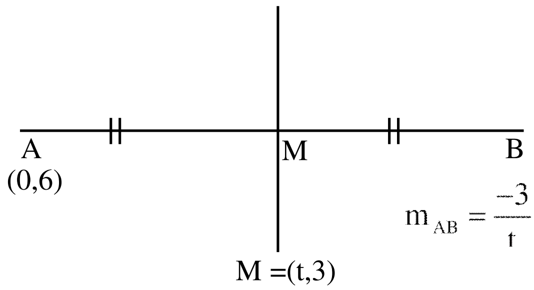
3. Let A be a fixed point (0, 6) and B be a moving point (2t, 0). Let M be the mid-point of AB and the perpendicular bisector of AB meets the y-axis at C. The locus of the mid-point P of MC is :

(1) $3x^2 - 2y - 6 = 0$ (2) $3x^2 + 2y - 6 = 0$

(3) $2x^2 + 3y - 9 = 0$ (4) $2x^2 - 3y + 9 = 0$

Official Ans. by NTA (3)

Sol. A(0,6) and B(2t,0)



Perpendicular bisector of AB is

$$(y-3) = \frac{t}{3}(x-t)$$

So, $C = \left(0, 3 - \frac{t^2}{3}\right)$

Let P be (h,k)

$$h = \frac{t}{2}; k = \left(3 - \frac{t^2}{6}\right)$$

$$\Rightarrow k = 3 - \frac{4h^2}{6} \Rightarrow 2x^2 + 3y - 9 = 0 \text{ option (3)}$$

4. If $(\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a$; $0 < x < 1$, $a \neq 0$, then the value of $2x^2 - 1$ is :

(1) $\cos\left(\frac{4a}{\pi}\right)$ (2) $\sin\left(\frac{2a}{\pi}\right)$

(3) $\cos\left(\frac{2a}{\pi}\right)$ (4) $\sin\left(\frac{4a}{\pi}\right)$

Official Ans. by NTA (2)

Sol. Given $a = (\sin^{-1}x)^2 - (\cos^{-1}x)^2$

$= (\sin^{-1}x + \cos^{-1}x)(\sin^{-1}x - \cos^{-1}x)$

$= \frac{\pi}{2}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$

$\Rightarrow 2\cos^{-1}x = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow 2x^2 - 1 = \cos\left(\frac{\pi}{2} - \frac{2a}{\pi}\right)$ option (2)

5. If the matrix $A = \begin{pmatrix} 0 & 2 \\ K & -1 \end{pmatrix}$ satisfies $A(A^3 + 3I) = 2I$, then the value of K is :

(1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) -1 (4) 1

Official Ans. by NTA (1)

Sol. Given matrix $A = \begin{bmatrix} 0 & 2 \\ k & -1 \end{bmatrix}$

$A^4 + 3IA = 2I$

$\Rightarrow A^4 = 2I - 3A$

Also characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 0-\lambda & 2 \\ k & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda + \lambda^2 - 2k = 0$

$\Rightarrow A + A^2 = 2K.I$

$\Rightarrow A^2 = 2KI - A$

$\Rightarrow A^4 = 4K^2I + A^2 - 4AK$

Put $A^2 = 2KI - A$

and $A^4 = 2I - 3A$

$2I - 3A = 4K^2I + 2KI - A - 4AK$

$\Rightarrow I(2 - 2K - 4K^2) = A(2 - 4K)$

$\Rightarrow -2I(2K^2 + K - 1) = 2A(1 - 2K)$

$\Rightarrow -2I(2K - 1)(K + 1) = 2A(1 - 2K)$

$\Rightarrow (2K - 1)(2A) - 2I(2K - 1)(K + 1) = 0$

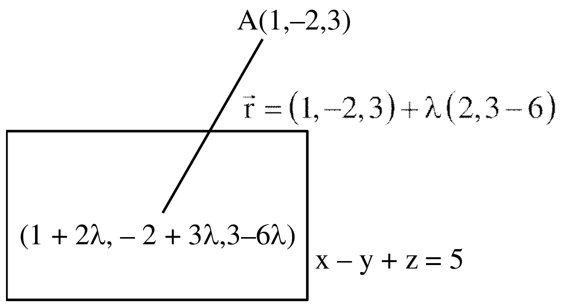
$\Rightarrow (2K - 1)[2A - 2I(K + 1)] = 0$

$\Rightarrow K = \frac{1}{2}$

6. The distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to a line, whose direction ratios are 2, 3, –6 is :

(1) 3 (2) 5 (3) 2 (4) 1

Official Ans. by NTA (4)



Sol.

$(1 + 2\lambda) + 2 - 3\lambda + 3 - 6\lambda = 5$

$\Rightarrow 6 - 7\lambda = 5 \Rightarrow \lambda = \frac{1}{7}$

so, $P = \left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$

$AP = \sqrt{\left(1 - \frac{9}{7}\right)^2 + \left(-2 + \frac{11}{7}\right)^2 + \left(3 - \frac{15}{7}\right)^2}$

$AP = \sqrt{\left(\frac{4}{49}\right) + \frac{9}{49} + \frac{36}{49}} = 1$

7. If $S = \left\{z \in \mathbb{C} : \frac{z - i}{z + 2i} \in \mathbb{R}\right\}$, then :

(1) S contains exactly two elements

(2) S contains only one element

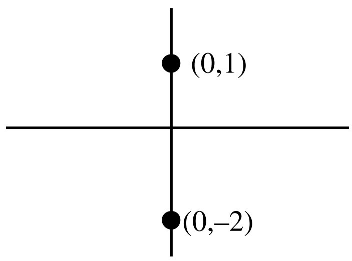
(3) S is a circle in the complex plane

(4) S is a straight line in the complex plane

Official Ans. by NTA (4)

Sol. Given $\frac{z - i}{z + 2i} \in R$

Then $\arg\left(\frac{z - i}{z + 2i}\right)$ is 0 or Π



$\Rightarrow$ S is straight line in complex

8. Let y = y(x) be the solution of the differential equation $\frac{dy}{dx} = 2(y + 2\sin x - 5)\,x - 2\cos x$ such that y(0) = 7. Then y(π) is equal to :

(1) $2e^{\pi^2} + 5$ (2) $e^{\pi^2} + 5$

(3) $3e^{\pi^2} + 5$ (4) $7e^{\pi^2} + 5$

Official Ans. by NTA (1)

Sol. $\frac{dy}{dx} - 2xy = 2(2\sin x - 5)x - 2\cos x$

$IF = e^{-x^2}$

so, $y.e^{-x^2} = \int e^{-x^2}(2x(2\sin x - 5) - 2\cos x)dx$

$\Rightarrow y.e^{-x^2} = e^{-x^2}(5 - 2\sin x) + c$

$\Rightarrow y = 5 - 2\sin x + c.e^{x^2}$

Given at x = 0, y = 7

$\Rightarrow 7 = 5 + c \Rightarrow c = 2$

So, $y = 5 - 2\sin x + 2e^{x^2}$

Now at x = π,

$y = 5 + 2e^{\pi^2}$

9. Equation of a plane at a distance $\sqrt{\frac{2}{21}}$ from the origin, which contains the line of intersection of the planes x – y – z – 1 = 0 and 2x + y – 3z + 4 = 0, is :

(1) 3x – y – 5z + 2 = 0 (2) 3x – 4z + 3 = 0

(3) –x + 2y + 2z – 3 = 0 (4) 4x – y – 5z + 2 = 0

Official Ans. by NTA (4)

Sol. Required equation of plane

$P_1 + \lambda P_2 = 0$

(x – y – z – 1) + λ(2x + y – 3z + 4) = 0

Given that its dist. From origin is $\frac{2}{\sqrt{21}}$

Thus $\frac{|4\lambda - 1|}{\sqrt{(2\lambda+1)^2 + (\lambda-1)^2 + (-3\lambda-1)^2}} = \frac{\sqrt{2}}{\sqrt{21}}$

$\Rightarrow 21(4\lambda - 1)^2 = 2(14\lambda^2 + 8\lambda + 3)$

$\Rightarrow 336\lambda^2 - 168\lambda + 21 = 28\lambda^2 + 16\lambda + 6$

$\Rightarrow 308\lambda^2 - 184\lambda + 15 = 0$

$\Rightarrow 308\lambda^2 - 154\lambda - 30\lambda + 15 = 0$

$\Rightarrow (2\lambda - 1)(154\lambda - 15) = 0$

$\Rightarrow \lambda = \frac{1}{2}$ or $\frac{15}{154}$

for $\lambda = \frac{1}{2}$ reqd. plane is

4x – y – 5z + 2 = 0

10. If $U_n = \left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)^2 \ldots \left(1 + \frac{n^2}{n^2}\right)^n$, then $\lim_{n\to\infty}(U_n)^{\frac{-4}{n^2}}$ is equal to :

(1) $\frac{e^2}{16}$ (2) $\frac{4}{e}$ (3) $\frac{16}{e^2}$ (4) $\frac{4}{e^2}$

Official Ans. by NTA (1)

Sol. $U_n = \prod_{r=1}^{n}\left(1 + \frac{r^2}{n^2}\right)^r$

$L = \lim_{n\to\infty}(U_n)^{-4/n^2}$

$\log L = \lim_{n\to\infty}\frac{-4}{n^2}\sum_{r=1}^{n}\log\left(1 + \frac{r^2}{n^2}\right)^r$

$\Rightarrow \log L = \lim_{n\to\infty}\sum_{r=1}^{n} -\frac{4r}{n}.\frac{1}{n}\log\left(1 + \frac{r^2}{n^2}\right)$

$\Rightarrow \log L \Rightarrow -4\int_0^1 x\log(1 + x^2)dx$

put $1 + x^2 = t$

Now, 2xdx = dt

$= -2\int_1^2 \log(t)dt = -2[t\log t - t]_1^2$

$\Rightarrow \log L = -2(2\log 2 - 1)$

$\therefore L = e^{-2(2\log 2 - 1)}$

$= e^{-2\left(\log\left(\frac{4}{e}\right)\right)}$

$= e^{\log\left(\frac{4}{e}\right)^{-2}}$

$= \left(\frac{e}{4}\right)^2 = \frac{e^2}{16}$

11. The statement $(p \wedge (p \to q) \wedge (q \to r)) \to r$ is :

(1) a tautology

(2) equivalent to $p \to \sim r$

(3) a fallacy

(4) equivalent to $q \to \sim r$

Official Ans. by NTA (1)

Sol. $(p \wedge (p \to q) \wedge (q \to r)) \to r$

$\equiv (p \wedge (\sim p \vee q) \vee (\sim q \vee r)) \to r$

$\equiv ((p \wedge q) \wedge (\sim p \vee r)) \to r$

$\equiv (p \wedge q \wedge r) \to r$

$\equiv \sim (p \wedge q \wedge r) \vee r$

$\equiv (\sim p) \vee (\sim q) \vee (\sim r) \vee r$

$\Rightarrow$ tautology

12. Let us consider a curve, $y = f(x)$ passing through the point $(-2, 2)$ and the slope of the tangent to the curve at any point $(x, f(x))$ is given by $f(x) + xf'(x) = x^2$. Then :

(1) $x^2 + 2xf(x) - 12 = 0$

(2) $x^3 + xf(x) + 12 = 0$

(3) $x^3 - 3xf(x) - 4 = 0$

(4) $x^2 + 2xf(x) + 4 = 0$

Official Ans. by NTA (3)

Sol. $y + \frac{xdy}{dx} = x^2$ (given)

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x$

If $= e^{\int \frac{1}{x}dx} = x$

Solution of DE

$\Rightarrow y.x = \int x.x\ dx$

$\Rightarrow xy = \frac{x^3}{3} + \frac{c}{3}$

Passes through $(-2,2)$, so

$-12 = -8 + c \Rightarrow c = -4$

$\therefore\ 3xy = x^3 - 4$

ie. $3x.f(x) = x^3 - 4$

13. $\sum_{k=0}^{20} \left({}^{20}C_k\right)^2$ is equal to :

(1) ${}^{40}C_{21}$ (2) ${}^{40}C_{19}$ (3) ${}^{40}C_{20}$ (4) ${}^{41}C_{20}$

Official Ans. by NTA (3)

Sol. $\sum_{k=0}^{20} {}^{20}C_k . {}^{20}C_{20-k}$

sum of suffix is const. so summation will be

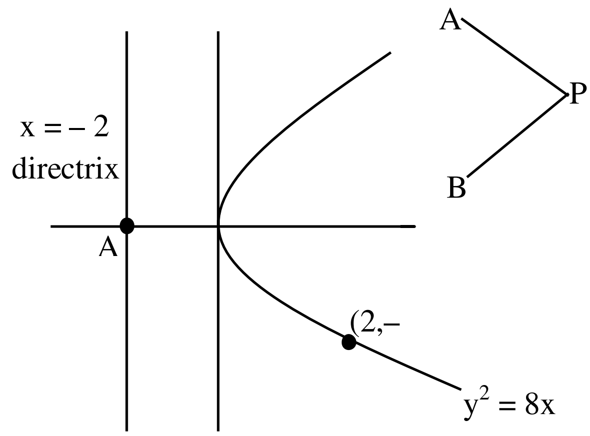
${}^{40}C_{20}$

14. A tangent and a normal are drawn at the point $P(2, -4)$ on the parabola $y^2 = 8x$, which meet the directrix of the parabola at the points A and B respectively. If $Q(a, b)$ is a point such that AQBP is a square, then $2a + b$ is equal to :

(1) –16 (2) –18 (3) –12 (4) –20

Official Ans. by NTA (1)

Sol.



Equation of tangent at $(2,-4)$ $(T = 0)$

$-4y = 4(x + 2)$

$x + y + 2 = 0$ ...(1)

equation of normal

$x - y + \lambda = 0$

$\downarrow(2,-4)$

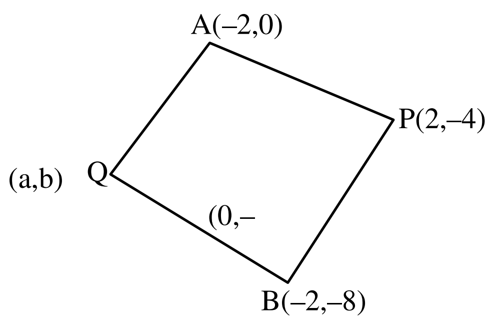
$\lambda = -6$

thus $x - y = 6$ ...(2) equation of normal

POI of (1) & $x = -2$ is $A(-2,0)$

POI of (2) & $x = -2$ is $A(-2,8)$

Given AQBP is a sq.



$\Rightarrow m_{AQ}.m_{AP} = -1$

$\Rightarrow \left(\frac{b}{a+2}\right)\left(\frac{4}{-4}\right) = -1 \Rightarrow a + 2 = b$ ...(1)

Also PQ must be parallel to x-axis thus

$\Rightarrow b = -4$

$\therefore\ a = -6$

Thus $2a + b = -16$

15. Let $\frac{\sin A}{\sin B}=\frac{\sin(A-C)}{\sin(C-B)}$, where A, B, C are angles of a triangle ABC. If the lengths of the sides opposite these angles are a, b, c respectively, then :

(1) $b^2 - a^2 = a^2 + c^2$
(2) b^2, c^2, a^2 are in A.P.
(3) c^2, a^2, b^2 are in A.P.
(4) a^2, b^2, c^2 are in A.P.

Official Ans. by NTA (2)

Sol. $\frac{\sin A}{\sin B}=\frac{\sin(A-C)}{\sin(C-B)}$

As A,B,C are angles of triangle

$A + B + C = \pi$

$A = \pi - (B + C)$

So, $\sin A = \sin(B + C)$ ...(1)

Similarly $\sin B = \sin(A + C)$ ...(2)

From (1) and (2)

$\frac{\sin(B+C)}{\sin(A+C)}=\frac{\sin(A-C)}{\sin(C-B)}$

$\sin(C + B).\sin(C - B) = \sin(A - C)\sin(A + C)$

$\sin^2 C-\sin^2 B=\sin^2 A-\sin^2 C$

$\{\because \sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y\}$

$2\sin^2 C=\sin^2 A+\sin^2 B$

By sine rule

$2c^2 = a^2 + b^2$

$\Rightarrow b^2, c^2$ and a^2 are in A.P.

16. If α, β are the distinct roots of $x^2 + bx + c = 0$, then $\lim\limits_{x\to\beta}\frac{e^{2(x^2+bx+c)}-1-2(x^2+bx+c)}{(x-\beta)^2}$ is equal to:

(1) $b^2 + 4c$ (2) $2(b^2 + 4c)$
(3) $2(b^2 - 4c)$ (4) $b^2 - 4c$

Official Ans. by NTA (3)

Sol. $\lim\limits_{x\to\beta}\frac{e^{2(x^2+bx+c)}-1-2(x^2+bx+c)}{(x-\beta)^2}$

$\Rightarrow \lim\limits_{x\to\beta}\frac{1\left(1+\frac{2(x^2+bx+c)}{1!}+\frac{2^2(x^2+bx+c)^2}{2!}+...\right)-1-2(x^2+bx+c)}{(x-\beta)^2}$

$\Rightarrow \lim\limits_{x\to\beta}\frac{2(x^2+bx+1)^2}{(x-\beta)^2}$

$\Rightarrow \lim\limits_{x\to\beta}\frac{2(x-\alpha)^2(x-\beta)^2}{(x-\beta)^2}$

$\Rightarrow 2(\beta-\alpha)^2=2(b^2-4c)$

17. When a certain biased die is rolled, a particular face occurs with probability $\frac{1}{6}-x$ and its opposite face occurs with probability $\frac{1}{6}+x$. All other faces occur with probability $\frac{1}{6}$. Note that opposite faces sum to 7 in any die. If $0 < x < \frac{1}{6}$, and the probability of obtaining total sum = 7, when such a die is rolled twice, is $\frac{13}{96}$, then the value of x is:

(1) $\frac{1}{16}$ (2) $\frac{1}{8}$ (3) $\frac{1}{9}$ (4) $\frac{1}{12}$

Official Ans. by NTA (2)

Sol. Probability of obtaining total sum 7 = probability of getting opposite faces.

Probability of getting opposite faces

$=2\left[\left(\frac{1}{6}-x\right)\left(\frac{1}{6}+x\right)+\frac{1}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{1}{6}\right]$

$\Rightarrow 2\left[\left(\frac{1}{6}-x\right)\left(\frac{1}{6}+x\right)+\frac{1}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{1}{6}\right]=\frac{13}{96}$

(given)

$x=\frac{1}{8}$

18. If $x^2 + 9y^2 - 4x + 3 = 0$, $x, y \in \mathbb{R}$, then x and y respectively lie in the intervals:

(1) $\left[-\frac{1}{3},\frac{1}{3}\right]$ and $\left[-\frac{1}{3},\frac{1}{3}\right]$

(2) $\left[-\frac{1}{3},\frac{1}{3}\right]$ and [1, 3]

(3) [1, 3] and [1, 3]

(4) [1, 3] and $\left[-\frac{1}{3},\frac{1}{3}\right]$

Official Ans. by NTA (4)

Sol. $x^2 + 9y^2 - 4x + 3 = 0$

$(x^2 - 4x) + (9y^2) + 3 = 0$

$(x^2 - 4x + 4) + (9y^2) + 3 - 4 = 0$

$(x - 2)^2 + (3y)^2 = 1$

$\frac{(x-2)^2}{(1)^2}+\frac{y^2}{\left(\frac{1}{3}\right)^2}=1$ (equation of an ellipse).

As it is equation of an ellipse, x & y can vary inside the ellipse.

So, $x-2\in[-1,1]$ and $y\in\left[-\frac{1}{3},\frac{1}{3}\right]$

$x\in[1,3]$ $y\in\left[-\frac{1}{3},\frac{1}{3}\right]$

19. $\int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$ is equal to:

(1) 6 (2) 8

(3) 5 (4) 10

Official Ans. by NTA (3)

Sol. Let $I = \int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$

$I = \int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x-22)^2} dx \quad \ldots(1)$

We know

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ (king)

So $I = \int_6^{16} \frac{\log_e (22-x)^2}{\log_e (22-x)^2 + \log_e (22-(22-x))^2}$

$I = \int_0^{16} \frac{\log_e (22-x)^2}{\log_e x^2 + \log_e (22-x)^2} dx \quad \ldots(2)$

(1) + (2)

$2I = \int_6^{16} 1.dx = 10$

$I = 5$

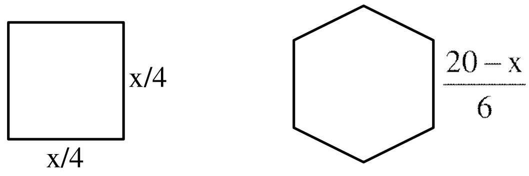
20. A wire of length 20 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a regular hexagon. Then the length of the side (in meters) of the hexagon, so that the combined area of the square and the hexagon is minimum, is:

(1) $\frac{5}{2+\sqrt{3}}$ (2) $\frac{10}{2+3\sqrt{3}}$

(3) $\frac{5}{3+\sqrt{3}}$ (4) $\frac{10}{3+2\sqrt{3}}$

Official Ans. by NTA (4)

Sol. Let the wire is cut into two pieces of length x and 20 – x.



Area of square $= \left(\frac{x}{4}\right)^2$ Area of regular hexagon

$= 6 \times \frac{\sqrt{3}}{4}\left(\frac{20-x}{6}\right)^2$

Total area $= A(x) = \frac{x^2}{16} + \frac{3\sqrt{3}}{2}\frac{(20-x)^2}{36}$

$A'(x) = \frac{2x}{16} + \frac{3\sqrt{3}\times 2}{2\times 36}(20-x)(-1)$

$A'(x) = 0$ at $x = \frac{40\sqrt{3}}{3+2\sqrt{3}}$

Length of side of regular Hexagon $= \frac{1}{6}(20-x)$

$= \frac{1}{6}\left(20 - \frac{4.\sqrt{3}}{3+2\sqrt{3}}\right)$

$= \frac{10}{2+2\sqrt{3}}$

SECTION-B

1. Let $\vec{a} = \hat{i} + 5\hat{j} + \alpha\hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} + \beta\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - 3\hat{k}$ be three vectors such that, $|\vec{b}\times\vec{c}| = 5\sqrt{3}$ and $\vec{a}$ is perpendicular to $\vec{b}$. Then the greatest amongst the values of $|\vec{a}|^2$ is _______.

Official Ans. by NTA (90)

Sol. since, $\vec{a}.\vec{b} = 0$

$1 + 15 + \alpha\beta = 0 \Rightarrow \alpha\beta = -16 \quad \ldots(1)$

Also,

$|\vec{b}\times\vec{c}|^2 = 75 \Rightarrow (10+\beta^2)14 - (5-3\beta)^2 = 75$

$\Rightarrow 5\beta^2 + 30\beta + 40 = 0$

$\Rightarrow \beta = -4, -2$

$\Rightarrow \alpha = 4, 8$

$\Rightarrow |\vec{a}|^2_{max} = (26+\alpha^2)_{max} = 90$

2. The number of distinct real roots of the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$ is __________.

Official Ans. by NTA (4)

Sol. $3x^4 + 4x^3 - 12x^2 + 4 = 0$

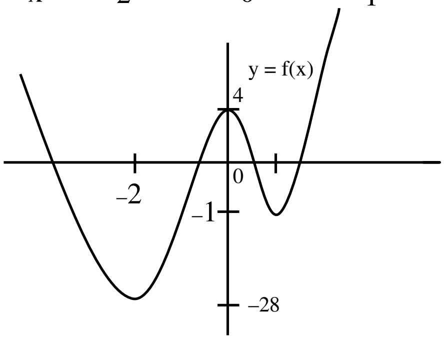
So, Let $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

$\therefore\ f'(x) = 12x(x^2 + x - 2)$

$= 12x\,(x + 2)\,(x - 1)$

$f'(x) = -$ (for $x < -2$), $+$ (for $-2 < x < 0$), $-$ (for $0 < x < 1$), $+$ (for $x > 1$)

x= -2 $\quad 0 \quad 1$



3. Let the equation $x^2 + y^2 + px + (1 - p)y + 5 = 0$ represent circles of varying radius $r \in (0, 5]$. Then the number of elements in the set $S = \{q : q = p^2$ and q is an integer$\}$ is __________.

Official Ans. by NTA (61)

Sol. $r = \sqrt{\dfrac{p^2}{4} + \dfrac{(1-p)^2}{4} - 5} = \dfrac{\sqrt{2p^2 - 2p - 19}}{2}$

Since, $r \in (0,5]$

So, $0 < 2p^2 - 2p - 19 \le 100$

$\Rightarrow p \in \left[\dfrac{1-\sqrt{239}}{2}, \dfrac{1-\sqrt{39}}{2}\right) \cup \left(\dfrac{1+\sqrt{39}}{2}, \dfrac{1+\sqrt{239}}{2}\right]$ so, number of integral values of p^2 is 61

4. If $A = \{x \in \mathbf{R} : |x - 2| > 1\}$, $B = \{x \in \mathbf{R} : \sqrt{x^2 - 3} > 1\}$, $C = \{x \in \mathbf{R} : |x - 4| \ge 2\}$ and $\mathbf{Z}$ is the set of all integers, then the number of subsets of the set $(A \cap B \cap C)^c \cap \mathbf{Z}$ is ____________.

Official Ans. by NTA (256)

Sol. $A = (-\infty, 1) \cup (3, \infty)$

$B = (-\infty, -2) \cup (2, \infty)$

$C = (-\infty, 2] \cup [6, \infty)$

So, $A \cap B \cap C = (-\infty, -2) \cup [6, \infty)$

$z \cap (A \cap B \cap C)' = \{-2, -1, 0, -1, 2, 3, 4, 5\}$

Hence no. of its subsets $= 2^8 = 256$.

5. If $\displaystyle\int \frac{dx}{(x^2 + x + 1)^2} = a\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + b\left(\frac{2x+1}{x^2+x+1}\right) + C$, $x > 0$ where C is the constant of integration, then the value of $9\left(\sqrt{3}a + b\right)$ is equal to ________.

Official Ans. by NTA (15)

Sol. $I = \displaystyle\int \frac{dx}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^2}$

$\displaystyle\int \frac{dt}{\left(t^2 + \frac{3}{4}\right)^2} \quad \left(\text{Put } x + \frac{1}{2} = t\right)$

$= \dfrac{\sqrt{3}}{2} \displaystyle\int \frac{\sec^2\theta\, d\theta}{\frac{9}{16}\sec^4\theta} \quad \left(\text{Put } t = \frac{\sqrt{3}}{2}\tan\theta\right)$

$= \dfrac{4\sqrt{3}}{9} \displaystyle\int (1 + \cos 2\theta)\, d\theta$

$= \dfrac{4\sqrt{3}}{9}\left[\theta + \dfrac{\sin 2\theta}{2}\right] + c$

$= \dfrac{4\sqrt{3}}{9}\left[\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + \dfrac{\sqrt{3}(2x+1)}{3 + (2x+1)^2}\right] + c$

$= \dfrac{4\sqrt{3}}{9}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + \dfrac{1}{3}\left(\dfrac{2x+1}{x^2+x+1}\right) + c$

Hence, $9\left(\sqrt{3}a + b\right) = 15$

6. If the system of linear equations

$2x + y - z = 3$

$x - y - z = \alpha$

$3x + 3y + \beta z = 3$

has infinitely many solution, then $\alpha + \beta - \alpha\beta$ is equal to __________.

Official Ans. by NTA (5)

Sol. $2 \times$ (i) – (ii) – (iii) gives :

$-(1 + \beta)z = 3 - \alpha$

For infinitely many solution

$\beta + 1 = 0 = 3 - \alpha \Rightarrow (\alpha, \beta) = (3, -1)$

Hence, $\alpha + \beta - \alpha\beta = 5$

7. Let n be an odd natural number such that the variance of 1, 2, 3, 4, ..., n is 14. Then n is equal to ________.

Official Ans. by NTA (13)

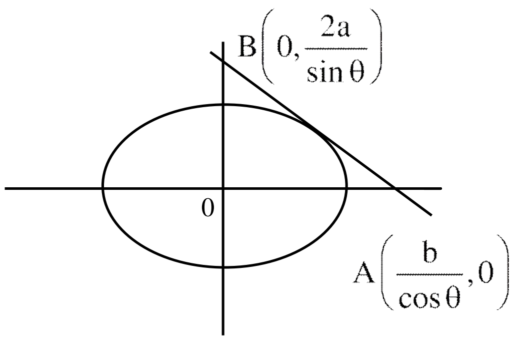
Sol. $\frac{n^2-1}{12}=14 \Rightarrow n=13$

8. If the minimum area of the triangle formed by a tangent to the ellipse $\frac{x^2}{b^2}+\frac{y^2}{4a^2}=1$ and the co-ordinate axis is kab, then k is equal to _______.

Official Ans. by NTA (2)

Sol. Tangent

$$\frac{x\cos\theta}{b}+\frac{y\sin\theta}{2a}=1$$



So, $\text{area}(\Delta OAB)=\frac{1}{2}\times\frac{b}{\cos\theta}\times\frac{2a}{\sin\theta}$

$$=\frac{2ab}{\sin 2\theta}\geq 2ab$$

$$\Rightarrow k=2$$

9. A number is called a palindrome if it reads the same backward as well as forward. For example 285582 is a six digit palindrome. The number of six digit palindromes, which are divisible by 55, is ________.

Official Ans. by NTA (100)

Sol.

5	a	b	b	a	5

It is always divisible by 5 and 11.

So, required number = 10 × 10 = 100

10. If $y^{1/4}+y^{-1/4}=2x$, and $(x^2-1)\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$, then $|\alpha-\beta|$ is equal to __________.

Official Ans. by NTA (17)

Sol. $y^{\frac{1}{4}}+\frac{1}{y^{\frac{1}{4}}}=2x$

$$\Rightarrow\left(y^{\frac{1}{4}}\right)^2-2xy^{\left(\frac{1}{4}\right)}+1=0$$

$$\Rightarrow y^{\frac{1}{4}}=x+\sqrt{x^2-1} \text{ or } x-\sqrt{x^2-1}$$

So, $\frac{1}{4}\frac{1}{y^{\frac{3}{4}}}\frac{dy}{dx}=1+\frac{x}{\sqrt{x^2-1}}$

$$\Rightarrow\frac{1}{4}\frac{1}{y^{3/4}}\frac{dy}{dx}=\frac{y^{\frac{1}{4}}}{\sqrt{x^2-1}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{4y}{\sqrt{x^2-1}} \quad \ldots(1)$$

Hence, $\frac{d^2y}{dx^2}=4\frac{\left(\sqrt{x^2-1}\right)y'-\frac{yx}{\sqrt{x^2-1}}}{x^2-1}$

$$\Rightarrow(x^2-1)y''=4\frac{(x^2-1)y'-xy}{\sqrt{x^2-1}}$$

$$\Rightarrow(x^2-1)y''=4\left(\sqrt{x^2-1}y'-\frac{xy}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow(x^2-1)y''=4\left(4y-\frac{xy'}{4}\right) \text{ (from I)}$$

$$\Rightarrow(x^2-1)y''+xy'-16y=0$$

So, $|\alpha-\beta|=17$