

FINAL JEE-MAIN EXAMINATION – SEPTEMBER, 2020

(Held On Wednesday 06th SEPTEMBER, 2020) TIME : 3 PM to 6 PM

MATHEMATICS

TEST PAPER WITH SOLUTION

1. The set of all real values of λ for which the function $f(x) = (1 - \cos^2 x)(\lambda + \sin x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, has exactly one maxima and exactly one minima, is :

- (1) $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$ (2) $\left(-\frac{1}{2}, \frac{1}{2}\right)$
 (3) $\left(-\frac{3}{2}, \frac{3}{2}\right)$ (4) $\left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

Official Ans. by NTA (4)

- Sol.** $f(x) = (1 - \cos^2 x)(\lambda + \sin x)$

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$f(x) = \lambda \sin^2 x + \sin^3 x$$

$$f'(x) = 2\lambda \sin x \cos x + 3\sin^2 x \cos x$$

$$f'(x) = \sin x \cos x (2\lambda + 3\sin x)$$

$$\sin x = 0, \frac{-2\lambda}{3}, (\lambda \neq 0)$$

for exactly one maxima & minima

$$\frac{-2\lambda}{3} \in (-1, 1) \Rightarrow \lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right)$$

$$\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$$

2. For all twice differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$, with $f(0) = f(1) = f'(0) = 0$

- (1) $f''(x) = 0$, for some $x \in (0, 1)$
 (2) $f''(0) = 0$
 (3) $f''(x) \neq 0$ at every point $x \in (0, 1)$
 (4) $f''(x) = 0$ at every point $x \in (0, 1)$

Official Ans. by NTA (1)

- Sol.** $f(0) = f(1) = f'(0) = 0$

Apply Rolles theorem on $y = f(x)$ in $x \in [0, 1]$

$$f(0) = f(1) = 0$$

$$\Rightarrow f'(\alpha) = 0 \text{ where } \alpha \in (0, 1)$$

Now apply Rolles theorem on $y = f'(x)$

in $x \in [0, \alpha]$

$f'(0) = f'(\alpha) = 0$ and $f'(x)$ is continuous and differentiable

$$\Rightarrow f''(\beta) = 0 \text{ for some } \beta \in (0, \alpha) \in (0, 1)$$

$$\Rightarrow f''(x) = 0 \text{ for some } x \in (0, 1)$$

3. If the tangent to the curve, $y = f(x) = x \log_e x$, ($x > 0$) at a point $(c, f(c))$ is parallel to the line - segment joining the points $(1, 0)$ and (e, e) , then c is equal to :

(1) $\frac{1}{e-1}$ (2) $e^{\left(\frac{1}{1-e}\right)}$

(3) $e^{\left(\frac{1}{e-1}\right)}$ (4) $\frac{e-1}{e}$

Official Ans. by NTA (3)

- Sol.** $f(x) = x \log_e x$

$$f'(x) \Big|_{(c, f(c))} = \frac{e-0}{e-1}$$

$$f'(x) = 1 + \log_e x$$

$$f'(x) \Big|_{(c, f(c))} = 1 + \log_e c = \frac{e}{e-1}$$

$$\log_e c = \frac{e-(e-1)}{e-1} = \frac{1}{e-1} \Rightarrow c = e^{\frac{1}{e-1}}$$

Sol. α and β are the roots of the equation $4x^2 + 2x - 1 = 0$

$$4\alpha^2 + 2\alpha = 1 \Rightarrow \frac{1}{2} = 2\alpha^2 + \alpha \quad \dots(1)$$

$$\beta = \frac{-1}{2} - \alpha$$

using equation (1)

$$\beta = -(2\alpha^2 + \alpha) - \alpha$$

$$\beta = -2\alpha^2 - 2\alpha$$

$$\beta = -2\alpha(\alpha + 1)$$

8. Let $z = x + iy$ be a non-zero complex number such that $z^2 = iz|z|^2$, where $i = \sqrt{-1}$, then z lies on the :

- (1) imaginary axis (2) real axis
 (3) line, $y = x$ (4) line, $y = -x$

Official Ans. by NTA (3)

Sol. $z = x + iy$
 $z^2 = iz|z|^2$

$$(x + iy)^2 = i(x^2 + y^2)$$

$$(x^2 - y^2) - i(x^2 + y^2 - 2xy) = 0$$

$$(x - y)(x + y) - i(x - y)^2 = 0$$

$$(x - y)((x + y) - i(x - y)) = 0$$

$$\Rightarrow x = y$$

z lies on $y = x$

9. The common difference of the A.P. b_1, b_2, \dots, b_m is 2 more than the common difference of A.P. a_1, a_2, \dots, a_n . If $a_{40} = -159$, $a_{100} = -399$ and $b_{100} = a_{70}$, then b_1 is equal to :

- (1) -127 (2) -81
 (3) 81 (4) 127

Official Ans. by NTA (2)

Sol. $a_1, a_2, \dots, a_n \rightarrow (CD = d)$
 $b_1, b_2, \dots, b_m \rightarrow (CD = d + 2)$

$$a_{40} = a + 39d = -159 \quad \dots(1)$$

$$a_{100} = a + 99d = -399 \quad \dots(2)$$

$$\text{Subtract : } 60d = -240 \Rightarrow d = -4$$

using equation (1)

$$a + 39(-4) = -159$$

$$a = 156 - 159 = -3$$

$$a_{70} = a + 69d = -3 + 69(-4) = -279 = b_{100}$$

$$b_{100} = -279$$

$$b_1 + 99(d + 2) = -279$$

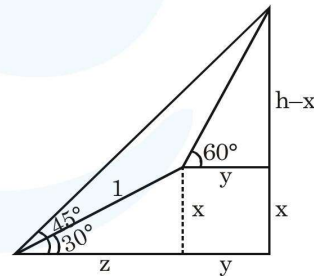
$$b_1 - 198 = -279 \Rightarrow b_1 = -81$$

10. The angle of elevation of the summit of a mountain from a point on the ground is 45° . After climbing up one km towards the summit at an inclination of 30° from the ground, the angle of elevation of the summit is found to be 60° . Then the height (in km) of the summit from the ground is :

- (1) $\frac{1}{\sqrt{3}-1}$ (2) $\frac{1}{\sqrt{3}+1}$
 (3) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (4) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

Official Ans. by NTA (1)

Sol.



$$\sin 30^\circ = x \Rightarrow x = \frac{1}{2}$$

$$\cos 30^\circ = z \Rightarrow z = \frac{\sqrt{3}}{2}$$

$$\tan 45^\circ = \frac{h}{y+z} \Rightarrow h = y + z$$

$$\tan 60^\circ = \frac{h-x}{y} \Rightarrow \tan 60^\circ = \frac{h-x}{h-z}$$

$$\sqrt{3}(h-z) = h-x$$

$$(\sqrt{3}-1)h = \sqrt{3}z-x$$

$$\Rightarrow (\sqrt{3}-1)h = \frac{3}{2} - \frac{1}{2}$$

$$\Rightarrow (\sqrt{3}-1)h = 1$$

$$h = \frac{1}{\sqrt{3}-1}$$

11. Let $\theta = \frac{\pi}{5}$ and $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. If $B = A$

+ A^4 , then $\det(B)$:

(1) is one (2) lies in (1, 2)

(3) is zero (4) lies in (2, 3)

Official Ans. by NTA (2)

Sol. $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

$$A^2 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$B = A + A^4$$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \begin{bmatrix} \cos 4\theta & \sin 4\theta \\ -\sin 4\theta & \cos 4\theta \end{bmatrix}$$

$$B = \begin{bmatrix} (\cos \theta + \cos 4\theta) & (\sin \theta + \sin 4\theta) \\ -(\sin \theta + \sin 4\theta) & (\cos \theta + \cos 4\theta) \end{bmatrix}$$

$$|B| = (\cos \theta + \cos 4\theta)^2 + (\sin \theta + \sin 4\theta)^2$$

$$|B| = 2 + 2\cos 3\theta, \quad \text{when } \theta = \frac{\pi}{5}$$

$$|B| = 2 + 2\cos \frac{3\pi}{5} = 2(1 - \sin 18)$$

$$|B| = 2 \left(1 - \frac{\sqrt{5}-1}{4} \right) = 2 \left(\frac{5-\sqrt{5}}{4} \right) = \frac{5-\sqrt{5}}{2}$$

12. For a suitably chosen real constant a, let a function, $f : \mathbb{R} - \{-a\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{a-x}{a+x}. \text{ Further suppose that for any real}$$

number $x \neq -a$ and $f(x) \neq -a$, $(f \circ f)(x) = x$. Then

$f\left(-\frac{1}{2}\right)$ is equal to :

(1) $\frac{1}{3}$ (2) 3

(3) -3 (4) $-\frac{1}{3}$

Official Ans. by NTA (2)

Sol. $f(x) = \frac{a-x}{a+x} \quad x \in \mathbb{R} - \{-a\} \rightarrow \mathbb{R}$

$$f(f(x)) = \frac{a-f(x)}{a+f(x)} = \frac{a - \left(\frac{a-x}{a+x}\right)}{a + \left(\frac{a-x}{a+x}\right)}$$

$$f(f(x)) = \frac{(a^2 - a) + x(a+1)}{(a^2 + a) + x(a-1)} = x$$

$$\Rightarrow (a^2 - a) + x(a+1) = (a^2 + a)x + x^2(a-1)$$

$$\Rightarrow a(a-1) + x(1-a^2) - x^2(a-1) = 0$$

$$\Rightarrow a = 1$$

$$f(x) = \frac{1-x}{1+x},$$

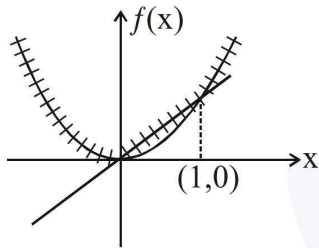
$$f\left(-\frac{1}{2}\right) = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = 3$$

13. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \max\{x, x^2\}$. Let S denote the set of all points in \mathbb{R} , where f is not differentiable. Then :

- (1) $\{0, 1\}$ (2) $\{0\}$
 (3) ϕ (an empty set) (4) $\{1\}$

Official Ans. by NTA (1)

Sol. $f(x) = \max(x, x^2)$



Non-differentiable at $x = 0, 1$

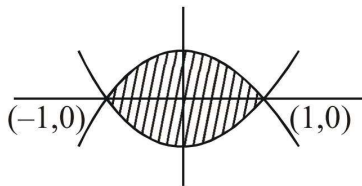
$S = \{0, 1\}$

14. The area (in sq. units) of the region enclosed by the curves $y = x^2 - 1$ and $y = 1 - x^2$ is equal to :

- (1) $\frac{4}{3}$ (2) $\frac{8}{3}$
 (3) $\frac{16}{3}$ (4) $\frac{7}{2}$

Official Ans. by NTA (2)

Sol. $y = x^2 - 1$ and $y = 1 - x^2$



$$A = \int_{-1}^1 ((1-x^2) - (x^2-1)) dx$$

$$A = \int_{-1}^1 (2-2x^2) dx = 4 \int_0^1 (1-x^2) dx$$

$$A = 4 \left(x - \frac{x^3}{3} \right)_0^1 = 4 \left(\frac{2}{3} \right) = \frac{8}{3}$$

15. The probabilities of three events A, B and C are given by $P(A) = 0.6, P(B) = 0.4$ and $P(C) = 0.5$. If $P(A \cup B) = 0.8, P(A \cap C) = 0.3, P(A \cap B \cap C) = 0.2, P(B \cap C) = \beta$ and $P(A \cup B \cup C) = \alpha$, where $0.85 \leq \alpha \leq 0.95$, then β lies in the interval:

- (1) $[0.36, 0.40]$ (2) $[0.35, 0.36]$
 (3) $[0.25, 0.35]$ (4) $[0.20, 0.25]$

Official Ans. by NTA (3)

Sol. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0.8 = 0.6 + 0.4 - P(A \cap B)$$

$$P(A \cap B) = 0.2$$

$$P(A \cup B \cup C) = \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C)$$

$$\alpha = 1.5 - (0.2 + 0.3 + \beta) + 0.2$$

$$\alpha = 1.2 - \beta \in [0.85, 0.95]$$

(where $\alpha \in [0.85, 0.95]$)

$$\beta \in [0.25, 0.35]$$

16. if the constant term in the binomial expansion

of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then $|k|$ equals :

- (1) 2 (2) 1
 (3) 3 (4) 9

Official Ans. by NTA (3)

Sol. $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$

$$T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \left(\frac{-k}{x^2}\right)^r$$

$$T_{r+1} = {}^{10}C_r \cdot x^{\frac{10-r}{2}} \cdot (-k)^r \cdot x^{-2r}$$

$$T_{r+1} = {}^{10}C_r x^{\frac{10-5r}{2}} (-k)^r$$

Constant term : $\frac{10-5r}{2} = 0 \Rightarrow r = 2$

$$T_3 = {}^{10}C_2 \cdot (-k)^2 = 405$$

$$k^2 = \frac{405}{45} = 9$$

$$k = \pm 3 \Rightarrow |k| = 3$$

17. The integral $\int_1^2 e^x \cdot x^x (2 + \log_e x) dx$ equal :

- (1) $e(4e + 1)$ (2) $e(2e - 1)$
 (3) $4e^2 - 1$ (4) $e(4e - 1)$

Official Ans. by NTA (4)

Sol. $\int_1^2 e^x \cdot x^x (2 + \log_e x) dx$

$$\int_1^2 e^x (2x^x + x^x \log_e x) dx$$

$$\int_1^2 e^x \left(\underbrace{x^x}_{f(x)} + \underbrace{x^x (1 + \log_e x)}_{f(x)} \right) dx$$

$$(e^x \cdot x^x)_1^2 = 4e^2 - e$$

18. Let L denote the line in the xy-plane with x and y intercepts as 3 and 1 respectively. Then the image of the point $(-1, -4)$ in this line is :

- (1) $\left(\frac{8}{5}, \frac{29}{5}\right)$ (2) $\left(\frac{29}{5}, \frac{11}{5}\right)$
 (3) $\left(\frac{11}{5}, \frac{28}{5}\right)$ (4) $\left(\frac{29}{5}, \frac{8}{5}\right)$

Official Ans. by NTA (3)

Sol. $L : \frac{x}{3} + \frac{y}{1} = 1 \Rightarrow x + 3y - 3 = 0$

Image of point $(-1, -4)$

$$\frac{x+1}{1} = \frac{y+4}{3} = -2 \left(\frac{-1-12-3}{10} \right)$$

$$\frac{x+1}{1} = \frac{y+4}{3} = \frac{16}{5}$$

$$(x, y) \equiv \left(\frac{11}{5}, \frac{28}{5} \right)$$

19. If $y = \left(\frac{2}{\pi}x - 1\right) \operatorname{cosec} x$ is the solution of the differential equation,

$$\frac{dy}{dx} + p(x)y = \frac{2}{\pi} \operatorname{cosec} x, \quad 0 < x < \frac{\pi}{2},$$

then the function $p(x)$ is equal to

- (1) $\cot x$ (2) $\tan x$
 (3) $\operatorname{cosec} x$ (4) $\sec x$

Official Ans. by NTA (1)

Sol. $y = \left(\frac{2x}{\pi} - 1\right) \operatorname{cosec} x \quad \dots(1)$

$$\frac{dy}{dx} = \frac{2}{\pi} \operatorname{cosec} x - \left(\frac{2x}{\pi} - 1\right) \operatorname{cosec} x \cot x$$

$$\frac{dy}{dx} = \frac{2 \operatorname{cosec} x}{\pi} - y \cot x$$

using equation (1)

$$\frac{dy}{dx} + y \cot x = \frac{2 \operatorname{cosec} x}{\pi}$$

$$\frac{dy}{dx} + p(x) \cdot y = \frac{2 \operatorname{cosec} x}{\pi} \quad x \in \left(0, \frac{\pi}{2}\right)$$

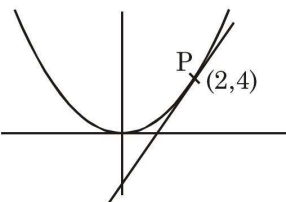
Compare : $p(x) = \cot x$

20. The centre of the circle passing through the point $(0, 1)$ and touching the parabola $y = x^2$ at the point $(2, 4)$ is :

- (1) $\left(\frac{3}{10}, \frac{16}{5}\right)$ (2) $\left(\frac{-16}{5}, \frac{53}{10}\right)$
 (3) $\left(\frac{6}{5}, \frac{53}{10}\right)$ (4) $\left(\frac{-53}{10}, \frac{16}{5}\right)$

Official Ans. by NTA (2)

Sol.



$y = x^2$
 $\left. \frac{dy}{dx} \right|_P = 4$
 $(y - 4) = 4(x - 2)$
 $4x - y - 4 = 0$

Circle : $(x - 2)^2 + (y - 4)^2 + \lambda(4x - y - 4) = 0$
passes through $(0, 1)$

$$4 + 9 + \lambda(-5) = 0 \Rightarrow \lambda = \frac{13}{5}$$

$$\text{Circle : } x^2 + y^2 + x(4\lambda - 4) + y(-\lambda - 8) + (20 - 4\lambda) = 0$$

$$\text{Centre : } \left(2 - 2\lambda, \frac{\lambda + 8}{2} \right) \equiv \left(\frac{-16}{5}, \frac{53}{10} \right)$$

- 21.** The sum of distinct values of λ for which the system of equations

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$

$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$

$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0,$$

has non-zero solutions, is _____.

Official Ans. by NTA (3.00)

- Sol.** $(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$
 $(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$
 $2x + (3\lambda + 1)y + (3\lambda - 3)z = 0$

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3\lambda - 3 \end{vmatrix} = 0$$

$$R_1 \rightarrow R_1 - R_2 \text{ \& } R_2 \rightarrow R_2 - R_3$$

$$\begin{vmatrix} 0 & 3 - \lambda & \lambda - 3 \\ \lambda - 3 & \lambda - 3 & -2(\lambda - 3) \\ 2 & 3\lambda + 1 & 3\lambda - 3 \end{vmatrix} = 0$$

$$(\lambda - 3)^2 \begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & 3\lambda + 1 & 3\lambda - 3 \end{vmatrix} = 0$$

$$(\lambda - 3)^2 [(3\lambda + 1) + (3\lambda - 1)] = 0$$

$$6\lambda(\lambda - 3)^2 = 0 \Rightarrow \lambda = 0, 3$$

$$\text{Sum} = 3$$

- 22.** Suppose that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and

$$f(1) = 3. \text{ If } \sum_{i=1}^n f(i) = 363, \text{ then } n \text{ is equal to}$$

_____.

Official Ans. by NTA (5.00)

- Sol.** $f(x + y) = f(x)f(y)$

$$\text{put } x = y = 1 \quad f(2) = (f(1))^2 = 3^2$$

$$\text{put } x = 2, y = 1 \quad f(3) = (f(1))^3 = 3^3$$

⋮

$$\text{Similarly } f(x) = 3^x$$

$$\sum_{i=1}^n f(i) = 363 \Rightarrow \sum_{i=1}^n 3^i = 363$$

$$(3 + 3^2 + \dots + 3^n) = 363$$

$$\frac{3(3^n - 1)}{2} = 363$$

$$3^n - 1 = 242 \Rightarrow 3^n = 243$$

$$\Rightarrow n = 5$$

- 23.** The number of words (with or without meaning) that can be formed from all the letters of the word "LETTER" in which vowels never come together is _____.

Official Ans. by NTA (120.00)

- Sol.** LETTER

$$\text{vowels} = EE, \text{ consonant} = LTTR$$

$$_ L _ T _ T _ R _$$

$$\frac{4!}{2!} \times {}^5C_2 \times \frac{2!}{2!} = 12 \times 10 = 120$$

24. Consider the data on x taking the values $0, 2, 4, 8, \dots, 2^n$ with frequencies ${}^n C_0, {}^n C_1, {}^n C_2, \dots, {}^n C_n$ respectively. If the mean of this data is $\frac{728}{2^n}$, then n is equal to _____.

Official Ans. by NTA (6.00)

Sol.

x	0	2	4	8		2^n
f	${}^n C_0$	${}^n C_1$	${}^n C_2$	${}^n C_3$		${}^n C_n$

$$\text{Mean} = \frac{\sum x_i f_i}{\sum f_i} = \frac{\sum_{r=1}^n 2^r \cdot {}^n C_r}{\sum_{r=0}^n {}^n C_r}$$

$$\text{Mean} = \frac{(1+2)^n - {}^n C_0}{2^n} = \frac{728}{2^n}$$

$$\Rightarrow \frac{3^n - 1}{2^n} = \frac{728}{2^n}$$

$$\Rightarrow 3^n = 729 \Rightarrow n = 6$$

25. If \vec{x} and \vec{y} be two non-zero vectors such that $|\vec{x} + \vec{y}| = |\vec{x}|$ and $2\vec{x} + \lambda\vec{y}$ is perpendicular to \vec{y} , then the value of λ is _____.

Official Ans. by NTA (1.00)

Sol. $|\vec{x} + \vec{y}| = |\vec{x}|$

$$\sqrt{|\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x} \cdot \vec{y}} = |\vec{x}|$$

$$|\vec{y}|^2 + 2\vec{x} \cdot \vec{y} = 0 \quad \dots (1)$$

$$\text{Now } (2\vec{x} + \lambda\vec{y}) \cdot \vec{y} = 0$$

$$2\vec{x} \cdot \vec{y} + \lambda|\vec{y}|^2 = 0$$

from (1)

$$-|\vec{y}|^2 + \lambda|\vec{y}|^2 = 0$$

$$(\lambda - 1)|\vec{y}|^2 = 0$$

$$\text{given } |\vec{y}| \neq 0 \Rightarrow \lambda = 1$$