FINAL JEE–MAIN EXAMINATION – MARCH, 2021

(Held On Tuesday 16th March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

1. The maximum value of

$$f(x)=\begin{vmatrix} \sin^2 x & 1+\cos^2 x & \cos 2x \\ 1+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}, x\in R \text{ is:}$$

(1) $\sqrt{7}$ (2) $\frac{3}{4}$ (3) $\sqrt{5}$ (4) 5

Official Ans by NTA (3)

Sol. $C_1 + C_2 \to C_1$

$$\begin{vmatrix} 2 & 1+\cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$R_1 - R_2 \to R_1$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

Open w.r.t. R_1

$-(2 \sin 2x - \cos 2x)$

$\cos 2x - 2 \sin 2x = f(x)$

$$f(x)\big|_{max} = \sqrt{1+4} = \sqrt{5}$$

2. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

(1) $\frac{9}{56}$ (2) $\frac{4}{9}$ (3) $\frac{3}{7}$ (4) $\frac{11}{27}$

Official Ans by NTA (2)

Sol. Total cases :

$\underline{6} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2}$

$n(s) = 6 \cdot 6!$

Favourable cases :

Number divisible by 3 $\equiv$

Sum of digits must be divisible by 3

Case-I

1, 2, 3, 4, 5, 6

Number of ways = 6!

Case-II

0, 1, 2, 4, 5, 6

Number of ways = $5 \cdot 5!$

Case-III

0, 1, 2, 3, 4, 5

Number of ways = $5 \cdot 5!$

n(favourable) = $6! + 2 \cdot 5 \cdot 5!$

$$P = \frac{6!+2\cdot 5\cdot 5!}{6\cdot 6!} = \frac{4}{9}$$

3. Let $\alpha \in R$ be such that the function

$$f(x)=\begin{cases} \dfrac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}, & x\neq 0 \\ \alpha, & x=0 \end{cases}$$

is continuous at x = 0, where $\{x\} = x - [x]$, $[x]$ is the greatest integer less than or equal to x. Then :

(1) $\alpha = \frac{\pi}{\sqrt{2}}$ (2) $\alpha = 0$

(3) no such α exists (4) $\alpha = \frac{\pi}{4}$

Official Ans by NTA (3)

Sol.

$$\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-}(x)$$

$$\lim_{x\to 0^+} \frac{\cos^{-1}(1-x^2)\cdot \sin^{-1}(1-x)}{x(1-x)(1+x)}$$

$$\lim_{x\to 0^+} \frac{\cos^{-1}(1-x^2)}{x\cdot 1\cdot 1}\cdot\frac{\pi}{2}$$

Let $1 - x^2 = \cos\theta$

$$\frac{\pi}{2}\lim_{x\to 0^+}\frac{\theta}{\sqrt{1-\cos\theta}}$$

$$\frac{\pi}{2}\lim_{\theta\to 0^+}\frac{\theta}{\sqrt{2}\sin\frac{\theta}{2}} = \frac{\pi}{\sqrt{2}}$$

Now, $\lim\limits_{x\to 0^-}\dfrac{\cos^{-1}\left(1-(1+x)^2\right)\sin^{-1}(-x)}{(1+x)-(1+x)^3}$

$$\lim_{x\to 0^-}\frac{\frac{\pi}{2}\left(-\sin^{-1}x\right)}{(1+x)(2+x)(-x)}$$

$$\lim_{x\to 0^-}\frac{\frac{\pi}{2}}{1\cdot 2}\cdot\frac{\sin^{-1}x}{x} = \frac{\pi}{4}$$

$\Rightarrow$ RHL $\neq$ LHL

Function can't be continuous

$\Rightarrow$ No value of α exist

4. If (x, y, z) be an arbitrary point lying on a plane P which passes through the point (42, 0, 0), (0, 42, 0) and (0, 0, 42), then the value of expression

$$3+\frac{x-11}{(y-19)^2(z-12)^2}+\frac{y-19}{(x-11)^2(z-12)^2}$$

$$+\frac{z-12}{(x-11)^2(y-19)^2}-\frac{x+y+z}{14(x-11)(y-19)(z-12)}$$

(1) 0 (2) 3 (3) 39 (4) –45

Official Ans by NTA (2)

Sol. Plane passing through (42, 0, 0), (0, 42, 0), (0, 0, 42)

From intercept from, equation of plane is

$x + y + z = 42$

$\Rightarrow (x - 11) + (y - 19) + (z - 12) = 0$

let $a = x - 11,\ b = y - 19,\ c = z - 12$

$a + b + c = 0$

Now, given expression is

$$3+\frac{a}{b^2c^2}+\frac{b}{a^2c^2}+\frac{c}{a^2b^2}-\frac{42}{14abc}$$

$$3+\frac{a^3+b^3+c^3-3abc}{a^2b^2c^2}$$

If $a + b + c = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

$\Rightarrow 3$

5. Consider the integral

$$I=\int_0^{10}\frac{[x]e^{[x]}}{e^{x-1}}dx,$$

where [x] denotes the greatest integer less than or equal to x. Then the value of I is equal to:

(1) $9(e - 1)$ (2) $45(e + 1)$

(3) $45(e - 1)$ (4) $9(e + 1)$

Official Ans by NTA (3)

Sol. $I=\int_0^{10}[x]\cdot e^{[x]-x+1}$

$$I=\int_0^1 0dx+\int_1^2 1\cdot e^{2-x}+\int_2^3 2\cdot e^{3-x}+.....+\int_9^{10}9\cdot e^{10-x}dx$$

$$\Rightarrow\ I=\sum_{n=0}^{9}\int_n^{n+1}n\cdot e^{n+1-x}dx$$

$$=-\sum_{n=0}^{9}n\left(e^{n+1-x}\right)_n^{n+1}$$

$$=-\sum_{n=0}^{9}n\cdot(e^0-e^1)$$

$$=(e-1)\sum_{n=0}^{9}n$$

$$=(e-1)\cdot\frac{9\cdot 10}{2}$$

$= 45(e - 1)$

6. Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line $y = x$. Then the equation of tangent to C at P(2,1) is :

(1) $x - y = 1$ (2) $2x + y = 5$
(3) $x + 3y = 5$ (4) $x + 2y = 4$

Official Ans by NTA (1)

Sol. Given $y^2 = 4x$

Mirror image on $y = x \Rightarrow$ C : $x^2 = 4y$

$$2x = 4\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\left.\frac{dy}{dx}\right|_{P(2,1)} = \frac{2}{2} = 1$$

Equation of tangent at (2, 1)

$\Rightarrow$ $y - 1 = 1(x - 2)$

$\Rightarrow$ $x - y = 1$

7. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + (\tan x)\,y = \sin x,\ 0 \le x \le \frac{\pi}{3}$, with $y(0) = 0$, then $y\left(\frac{\pi}{4}\right)$ equal to :

(1) $\frac{1}{4}\log_e 2$ (2) $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$

(3) $\log_e 2$ (4) $\frac{1}{2}\log_e 2$

Official Ans by NTA (2)

Sol. $\frac{dy}{dx} + (\tan x)y = \sin x\ ;\ 0 \le x \le \frac{\pi}{3}$

$$\text{I.F.} = e^{\int \tan x\,dx} = e^{\ell n \sec x} = \sec x$$

$$y \sec x = \int \tan x\,dx$$

$$y \sec x = \int \tan x dx$$

$$y \sec x = \ell n\,|\sec x| + C$$

$x = 0,\ y = 0 \quad \Rightarrow \quad \therefore\ c = 0$

$$y \sec x = \ell n\,|\sec x|$$

$$y = \cos x \cdot \ell n\,|\sec x|$$

$$y|_{x=\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}}\right)\cdot \ell n\sqrt{2}$$

$$y|_{x=\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}\log_e 2$$

8. Let A = {2, 3, 4, 5, ... , 30} and '$\simeq$' be an equivalence relation on A × A, defined by (a, b) $\simeq$ (c, d), if and only if ad = bc. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to :

(1) 5 (2) 6 (3) 8 (4) 7

Official Ans by NTA (4)

Sol. A = {2, 3, 4, 5, ... , 30}

(a, b) $\simeq$ (c, d) $\Rightarrow$ ad = bc

(4, 3) $\simeq$ (c, d) $\Rightarrow$ 4d = 3c

$$\Rightarrow \frac{4}{3} = \frac{c}{d}$$

$\frac{c}{d} = \frac{4}{3}$ & c, d $\in$ {2, 3, ..., 30}

(c, d) = {(4, 3), (8, 6), (12, 9), (16, 12), (20, 15), (24, 18), (28, 21)}

No. of ordered pair = 7

9. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax + 2ay + c = 0$, $(a < 0)$ be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line $x + 2y = 0$, is euqal to :

(1) $\sqrt{11}$ (2) $\sqrt{7}$ (3) $\sqrt{6}$ (4) $\sqrt{10}$

Official Ans by NTA (3)

Sol. $x^2 + y^2 + ax + 2ay + c = 0$

$$2\sqrt{g^2 - c} = 2\sqrt{\frac{a^2}{4} - c} = 2\sqrt{2}$$

$$\Rightarrow \frac{a^2}{4} - c = 2 \qquad \ldots(1)$$

$$2\sqrt{f^2-c}=2\sqrt{a^2-c}=2\sqrt{5}$$

$\Rightarrow\ a^2 - c = 5$ ...(2)

(1) & (2)

$\frac{3a^2}{4}=3 \Rightarrow a = -2 \quad (a < 0)$

$\therefore\ c = -1$

Circle $\Rightarrow x^2 + y^2 - 2x - 4y - 1 = 0$

$\Rightarrow (x-1)^2 + (y-2)^2 = 6$

Given $x + 2y = 0 \Rightarrow m = -\frac{1}{2}$

$m_{tangent} = 2$

Equation of tangent

$\Rightarrow (y-2) = 2(x-1) \pm \sqrt{6}\sqrt{1+4}$

$\Rightarrow 2x - y \pm \sqrt{30} = 0$

Perpendicular distance from (0, 0) $= \left|\frac{\pm\sqrt{30}}{\sqrt{4+1}}\right| = \sqrt{6}$

10. The least value of $|z|$ where z is complex number which satisfies the inequality

$$\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|}\log_e 2\right) \ge \log_{\sqrt{2}}\left|5\sqrt{7}+9i\right|,$$

$i=\sqrt{-1}$, is equal to :

(1) 3 (2) $\sqrt{5}$ (3) 2 (4) 8

Official Ans by NTA (1)

Sol. $\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|}\ell n\, 2\right) \ge \log_{\sqrt{2}}\left|5\sqrt{7}+9i\right|$

$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \ge \log_{\sqrt{2}}(16)$

$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \ge 2^3$

$\Rightarrow \frac{(|z|+3)(|z|-1)}{(|z|+1)} \ge 3$

$\Rightarrow (|z| + 3)(|z| - 1) \ge 3(|z| + 1)$

$|z|^2 + 2|z| - 3 \ge 3|z| + 3$

$\Rightarrow |z|^2 + |z| - 6 \ge 0$

$\Rightarrow (|z| - 3)(|z| + 2) \ge 0 \Rightarrow |z| - 3 \ge 0$

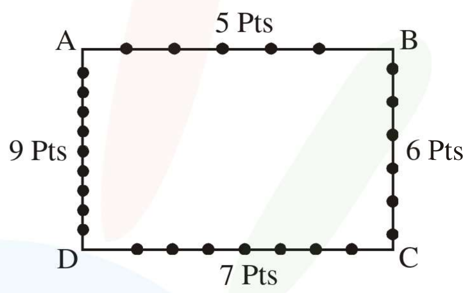
$\Rightarrow |z| \ge 3 \quad \Rightarrow |z|_{min} = 3$

11. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to :

(1) 795 (2) 1173 (3) 1890 (4) 717

Official Ans by NTA (4)

Sol.



α = Number of triangles

$\alpha = 5{\cdot}6{\cdot}7 + 5{\cdot}7{\cdot}9 + 5{\cdot}6{\cdot}9 + 6{\cdot}7{\cdot}9$

$= 210 + 315 + 270 + 378$

$= 1173$

β = Number of Quadrilateral

$\beta = 5{\cdot}6{\cdot}7{\cdot}9 = 1890$

$\beta - \alpha = 1890 - 1173 = 717$

12. If the point of intersections of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = 4b$, $b > 4$ lie on the curve $y^2 = 3x^2$, then b is equal to:

(1) 12 (2) 5 (3) 6 (4) 10

Official Ans by NTA (1)

Sol. $y^2 = 3x^2$

and $x^2 + y^2 = 4b$

Solve both we get

so $x^2 = b$

$$\frac{x^2}{16}+\frac{3x^2}{b^2}=1$$

$$\frac{b}{16}+\frac{3}{b}=1$$

$b^2 - 16b + 48 = 0$

$(b - 12)(b - 4) = 0$

$b = 12,\ b > 4$

13. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right)+\sin^{-1}\left(\frac{4x}{5}\right)=\sin^{-1}x$ is equal to:

(1) 2 (2) 1 (3) 3 (4) 0

Official Ans by NTA (3)

Sol. $$\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$$

$$\sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right)=\sin^{-1}x$$

$$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}=x$$

$x = 0,\ 3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$

$4\sqrt{25-9x^2}=25-3\sqrt{25-16x^2}$ squaring we get

$16(25-9x^2) = 625+9(25-16x^2)-150\sqrt{25-16x^2}$

$400 = 625 + 225 - 150\sqrt{25-16x^2}$

$\sqrt{25-16x^2} = 3 \Rightarrow 25 - 16x^2 = 9$

$\Rightarrow x^2 = 1$

Put x = 0, 1, –1 in the original equation

We see that all values satisfy the original equation.

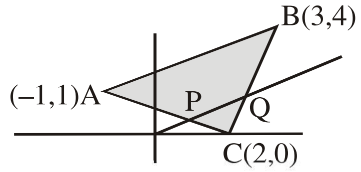
Number of solution = 3

14. Let A(–1, 1), B(3, 4) and C(2, 0) be given three points. A line y = mx, m > 0, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of ΔABC and ΔPQC respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

(1) $\frac{4}{15}$ (2) 1 (3) 2 (4) 3

Official Ans by NTA (2)

Sol.



$P \equiv (x_1, mx_1)$

$Q \equiv (x_2, mx_2)$

$$A_1=\frac{1}{2}\begin{vmatrix}3 & 4 & 1\\ 2 & 0 & 1\\ -1 & 1 & 1\end{vmatrix}=\frac{13}{2}$$

$$A_2=\frac{1}{2}\begin{vmatrix}x_1 & mx_1 & 1\\ x_2 & mx_2 & 1\\ 2 & 0 & 1\end{vmatrix}$$

$$A_2=\frac{1}{2}\left|2(mx_1-mx_2)\right|=m\left|x_1-x_2\right|$$

$$A_1 = 3A_2 \Rightarrow \frac{13}{2} = 3m\left|x_1-x_2\right|$$

$$\Rightarrow \left|x_1-x_2\right|=\frac{16}{6m}$$

AC : x + 3y = 2

BC : y = 4x – 8

$P : x + 3y = 2\ \&\ y = mx \Rightarrow x_1 = \frac{2}{1+3m}$

$Q : y = 4x - 8\ \&\ y = mx \Rightarrow x_2 = \frac{8}{4-m}$

$$\left|x_1-x_2\right|=\left|\frac{2}{1+3m}-\frac{8}{4-m}\right|$$

$$=\left|\frac{-26m}{(1+3m)(4-m)}\right| = \frac{26m}{(3m+1)|m-4|}$$

$$= \frac{26m}{(3m+1)(4-m)}$$

$$|x_1 - x_2| = \frac{13}{6m}$$

$$\frac{26m}{(3m+1)(4-m)} = \frac{13}{6m}$$

$\Rightarrow$ $12m^2 = -(3m + 1)(m - 4)$

$\Rightarrow$ $12m^2 = -(3m^2 - 11m - 4)$

$\Rightarrow$ $15m^2 - 11m - 4 = 0$

$\Rightarrow$ $15m^2 - 15m + 4m - 4 = 0$

$\Rightarrow$ $(15m + 4)(m - 1) = 0$

$\Rightarrow$ $m = 1$

15. Let f be a real valued function, defined on $R - \{-1, 1\}$ and given by

$$f(x) = 3\log_e\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1}.$$

Then in which of the following intervals, function f(x) is increasing?

(1) $(-\infty,-1)\cup\left(\left[\frac{1}{2},\infty\right)-\{1\}\right)$

(2) $(-\infty, \infty) - \{-1, 1\}$

(3) $\left(-1,\frac{1}{2}\right]$

(4) $\left(-\infty,\frac{1}{2}\right]-\{-1\}$

Official Ans by NTA (1)

Sol. $f(x) = 3\ell n(x-1) - 3\ell n(x+1) - \frac{2}{x-1}$

$$f'(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2}{(x-1)^2}$$

$$f'(x) = \frac{4(2x-1)}{(x-1)^2(x+1)}$$

$f'(x) \ge 0$

$$\Rightarrow x \in (-\infty,-1)\cup\left[\frac{1}{2},1\right)\cup(1,\infty)$$

16. Let $f : S \to S$ where $S = (0, \infty)$ be a twice differentiable function such that $f(x + 1) = xf(x)$. If $g : S \to R$ be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to :

(1) $\frac{205}{144}$ (2) $\frac{197}{144}$ (3) $\frac{187}{144}$ (4) 1

Official Ans by NTA (1)

Sol. $\ell nf(x + 1) = \ell n(xf(x))$

$\ell nf(x + 1) = \ell nx + \ell nf(x)$

$\Rightarrow$ $g(x + 1) = \ell nx + g(x)$

$\Rightarrow$ $g(x + 1) - g(x) = \ell nx$

$$\Rightarrow g''(x + 1) - g''(x) = -\frac{1}{x^2}$$

Put x = 1, 2, 3, 4

$$g''(2) - g''(1) = -\frac{1}{1^2} \quad ...(1)$$

$$g''(3) - g''(2) = -\frac{1}{2^2} \quad ...(2)$$

$$g''(4) - g''(3) = -\frac{1}{3^2} \quad ...(3)$$

$$g''(5) - g''(4) = -\frac{1}{4^2} \quad ...(4)$$

Add all the equation we get

$$g''(5) - g''(1) = -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$$

$$|g''(5) - g''(1)| = \frac{205}{144}$$

17. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)dx = 1$ and P(x) leaves remainder 5 when it is divided by $(x - 2)$. Then the value of $9(b + c)$ is equal to:

(1) 9 (2) 15 (3) 7 (4) 11

Official Ans by NTA (3)

Sol. $\int_0^1 (x^2+bx+c)dx = 1$

$\frac{1}{3}+\frac{b}{2}+c=1 \quad \Rightarrow \quad \frac{b}{2}+c=\frac{2}{3}$

$3b+6c=4$ ...(1)

P(2) = 5

4 + 2b + c = 5

2b + c = 1 ...(2)

From (1) & (2)

$b = \frac{2}{9}$ & $c = \frac{5}{9}$

9(b + c) = 7

18. If the foot of the perpendicular from point (4, 3, 8) on the line $L_1 : \frac{x-a}{l}=\frac{y-2}{3}=\frac{z-b}{4}$, $l \neq 0$ is (3, 5, 7), then the shortest distance between the line L_1 and line $L_2 : \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ is equal to :

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{6}}$ (3) $\sqrt{\frac{2}{3}}$ (4) $\frac{1}{\sqrt{3}}$

Official Ans by NTA (2)

Sol. (3,5,7) satisfy the line L_1

$\frac{3-a}{\ell}=\frac{5-2}{3}=\frac{7-b}{4}$

$\frac{3-a}{\ell}=1$ & $\frac{7-b}{4}=1$

$a+\ell=3$ ...(1) & $b=3$ ...(2)

$\vec{v}_1=<4,3,8>-<3,5,7>$

$\vec{v}_1=<1,-2,1>$

$\vec{v}_2=<\ell,3,4>$

$\vec{v}_1.\vec{v}_2=0 \Rightarrow \ell-6+4=0 \Rightarrow \ell=2$

$a+\ell=3 \Rightarrow a=1$

$L_1 : \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$

$L_2 : \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$

A = <1, 2, 3>

B = <2, 4, 5>

$\overrightarrow{AB}=<1,2,2>$

$\vec{p}=2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{q}=3\hat{i}+4\hat{j}+5\hat{k}$

$\vec{p}\times\vec{q}=-\hat{i}+2\hat{j}-\hat{k}$

Shortest distance = $\left|\frac{\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \frac{1}{\sqrt{6}}$

19. Let C_1 be the curve obtained by the solution of differential equation $2xy\frac{dy}{dx}=y^2-x^2$, x > 0. Let the curve C_2 be the solution of $\frac{2xy}{x^2-y^2}=\frac{dy}{dx}$. If both the curves pass through (1,1), then the area enclosed by the curves C_1 and C_2 is equal to :

(1) π – 1 (2) $\frac{\pi}{2}-1$ (3) π + 1 (4) $\frac{\pi}{4}+1$

Official Ans by NTA (2)

Sol. $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$, $x \in (0, \infty)$

put y = vx

$x\frac{dv}{dx}+v=\frac{v^2-1}{2v}$

$\frac{2v}{v^2+1}dv=-\frac{dx}{x}$

Integrate,

$\ln(v^2+1) = -\ln x + C$

$\ln\left(\frac{y^2}{x^2}+1\right)=-\ln x+C$

put x = 1, y = 1, C = $ln2$

$$ln\left(\frac{y^2}{x^2}+1\right)=-lnx+ln2$$

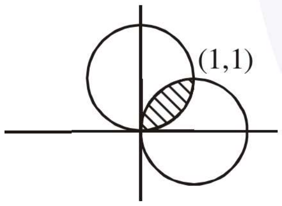
$\Rightarrow$ $x^2 + y^2 - 2x = 0$ (Curve C_1)

Similarly,

$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

Put y = vx

$x^2 + y^2 - 2y = 0$



required area = $2\int_0^1\left(\sqrt{2x-x^2}-x\right)dx = \frac{\pi}{2} - 1$

20. Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=2\hat{i}-3\hat{j}+5\hat{k}$. If $\vec{r}\times\vec{a}=\vec{b}\times\vec{r}$, $\vec{r}.\left(\alpha\hat{i}+2\hat{j}+\hat{k}\right)=3$ and $\vec{r}.\left(2\hat{i}+5\hat{j}-\alpha\hat{k}\right)=-1$, $\alpha \in R$, then the value of $\alpha+\left|\vec{r}\right|^2$ is equal to :

(1) 9 (2) 15 (3) 13 (4) 11

Official Ans by NTA (2)

Sol. $\vec{r}\times\vec{a}=\vec{b}\times\vec{r} \Rightarrow \vec{r}\times(\vec{a}+\vec{b})=0$

$\vec{r}=\vec{\lambda}(\vec{a}+\vec{b}) \Rightarrow \vec{r}=\vec{\lambda}(\hat{i}+2\hat{j}-3\hat{k}+2\hat{i}-3\hat{j}+5\hat{k})$

$\vec{r}=\vec{\lambda}(3\hat{i}-\hat{j}+2\hat{k})$ ...(1)

$\vec{r}\cdot(\alpha\hat{i}+2\hat{j}+\hat{k})=3$

Put $\vec{r}$ from (1) $\alpha\lambda=1$ ...(2)

$\vec{r}\cdot(2\hat{i}+5\hat{j}-\alpha\hat{k})=-1$

Put $\vec{r}$ from (1) $2\lambda\alpha-\lambda=1$ ...(3)

Solve (2) & (3)

$\alpha = 1, \lambda=1$

$\Rightarrow \vec{r}=3\hat{i}-\hat{j}+2\hat{k}$

$\left|\vec{r}\right|^2=14$ & $\alpha = 1$

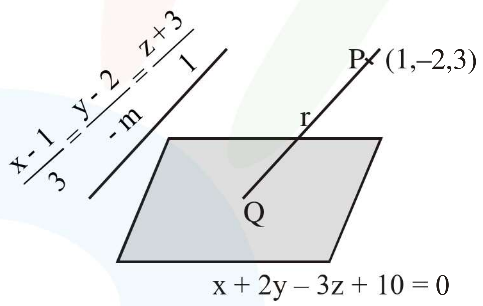
$\alpha+\left|\vec{r}\right|^2 = 15$

SECTION-B

1. If the distance of the point (1, –2, 3) from the plane x + 2y – 3z + 10 = 0 measured parallel to the line, $\frac{x-1}{3}=\frac{2-y}{m}=\frac{z+3}{1}$ is $\sqrt{\frac{7}{2}}$, then the value of |m| is equal to ______.

Official Ans by NTA (2)

Sol.



$$\text{DC of line} \equiv \left(\frac{3}{\sqrt{m^2+10}},\frac{-m}{\sqrt{m^2+10}},\frac{1}{\sqrt{m^2+10}}\right)$$

$$Q \equiv \left(1+\frac{3r}{\sqrt{m^2+10}},-2+\frac{-mr}{\sqrt{m^2+10}},3+\frac{r}{\sqrt{m^2+10}}\right)$$

Q lies on x + 2y – 3z + 10 = 0

$$1+\frac{3r}{\sqrt{m^2+10}}-4-\frac{2mr}{\sqrt{m^2+10}}-9-\frac{3r}{\sqrt{m^2+10}}+10=0$$

$$\Rightarrow \frac{r}{\sqrt{m^2+10}}(3-2m-3)=2$$

$$\Rightarrow \frac{r}{\sqrt{m^2+10}}(-2m)=2$$

$r^2m^2 = m^2 + 10$

$\frac{7}{2}m^2 = m^2 + 10 \Rightarrow \frac{5}{2}m^2 = 10 \Rightarrow m^2 = 4$

$|m| = 2$

2. Consider the statistics of two sets of observations as follows :

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value of n is equal to __________.

Official Ans by NTA (5)

Sol. $\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1+n_2)}(\bar{x}_1 - \bar{x}_2)^2$

$n_1 = 10,\ n_2 = n,\ \sigma_1^2 = 2,\ \sigma_2^2 = 1$

$\bar{x}_1 = 2,\ \bar{x}_2 = 3,\ \sigma^2 = \frac{17}{9}$

$\frac{17}{9} = \frac{10\times 2 + n}{n+10} + \frac{10n}{(n+10)^2}(3-2)^2$

$\Rightarrow \frac{17}{9} = \frac{(n+20)(n+10)+10n}{(n+10)^2}$

$\Rightarrow 17n^2 + 1700 + 340n = 90n + 9(n^2+30n+200)$

$\Rightarrow 8n^2 - 20n - 100 = 0$

$2n^2 - 5n - 25 = 0$

$\Rightarrow (2n+5)(n-5) = 0 \Rightarrow n = \underset{\text{(Rejected)}}{\frac{-5}{2}\Big\downarrow}, 5$

Hence n = 5

3. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two 2 × 1 matrices with real entries such that A = XB, where $X = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$, and $k \in R$. If $a_1^2 + a_2^2 = \frac{2}{3}\left(b_1^2 + b_2^2\right)$ and $(k^2+1)b_2^2 \neq -2b_1b_2$, then the value of k is _______.

Official Ans by NTA (1)

Sol. A = XB

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\begin{bmatrix} \sqrt{3}a_1 \\ \sqrt{3}a_2 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix}$

$b_1 - b_2 = \sqrt{3}a_1$ ...(1)

$b_1 + kb_2 = \sqrt{3}a_2$ ...(2)

Given, $a_1^2 + a_2^2 = \frac{2}{3}\left(b_1^2 + b_2^2\right)$

$(1)^2 + (2)^2$

$(b_1 + b_2)^2 + (b_1 + kb_2)^2 = 3(a_1^2 + a_2^2)$

$a_1^2 + a_2^2 = \frac{2}{3}b_1^2 + \frac{(1+k^2)}{3}b_2^2 + \frac{2}{3}b_1b_2(k-1)$

Given, $a_1^2 + a_2^2 = \frac{2}{3}b_1^2 + \frac{2}{3}b_2^2$

On comparing we get

$\frac{k^2+1}{3} = \frac{2}{3} \Rightarrow k^2 + 1 = 2$

$\Rightarrow k = \pm 1$ ....(3)

& $\frac{2}{3}(k-1) = 0 \Rightarrow k = 1$ ...(4)

From both we get k = 1

4. For real numbers α, β, γ and δ, if

$\int \frac{(x^2-1) + \tan^{-1}\left(\frac{x^2+1}{x}\right)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx$

$$= \alpha \log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right)$$

$$+\beta \tan^{-1}\left(\frac{\gamma(x^2-1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

where C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to __________.

Official Ans by NTA (6)

Sol. $$\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} + \int \frac{dx}{x^4+3x^2+1}$$

$$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)} + \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)dx}{x^4+3x^2+1}$$

Put $\tan^{-1}\left(x+\frac{1}{x}\right) = t$

$$\int \frac{dt}{t} + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$$

Put $x-\frac{1}{x} = y$, $x+\frac{1}{x} = z$

$$\log_e t + \frac{1}{2}\int \frac{dy}{y^2+5} - \frac{1}{2}\int \frac{dz}{z^2+1}$$

$$= \log_e \tan^{-1}\left(x+\frac{1}{x}\right) + \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)$$

$$-\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$\alpha = 1$, $\beta = \frac{1}{2\sqrt{5}}$, $\gamma = \frac{1}{\sqrt{5}}$, $\delta = \frac{-1}{2}$

or

$\alpha = 1$, $\beta = \frac{-1}{2\sqrt{5}}$, $\gamma = \frac{-1}{\sqrt{5}}$, $\delta = \frac{-1}{2}$

$$10(\alpha + \beta\gamma + \delta) = 10\left(1+\frac{1}{10}-\frac{1}{2}\right) = 6$$

5. Let $f : R \to R$ and $g : R \to R$ be defined as

$$f(x) = \begin{cases} x+a, & x<0 \\ |x-1|, & x \ge 0 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x+1, & x<0 \\ (x-1)^2+b, & x \ge 0 \end{cases}$$

where a, b are non-negative real numbers. If $(gof)(x)$ is continuous for all $x \in R$, then a + b is equal to ___________.

Official Ans by NTA (1)

Sol. $$g[f(x)] = \begin{bmatrix} f(x)+1 & f(x)<0 \\ (f(x)-1)^2+b & f(x) \ge 0 \end{bmatrix}$$

$$g[f(x)] = \begin{bmatrix} x+a+1 & x+a<0 \,\&\, x<0 \\ |x-1|+1 & |x-1|<0 \,\&\, x \ge 0 \\ (x+a-1)^2+b & x+a \ge 0 \,\&\, x<0 \\ (|x-1|-1)^2+b & |x-1| \ge 0 \,\&\, x \ge 0 \end{bmatrix}$$

$$g[f(x)] = \begin{bmatrix} x+a+1 & x \in (-\infty,-a) \,\&\, x \in (-\infty,0) \\ |x-1|+1 & x \in \phi \\ (x+a-1)^2+b & x \in [-a,\infty) \,\&\, x \in (-\infty,0) \\ (|x-1|-1)^2+b & x \in R \,\&\, x \in [0,\infty) \end{bmatrix}$$

$$g[f(x)] = \begin{bmatrix} x+a+1 & x \in (-\infty,-a) \\ (x+a-1)^2+b & x \in [-a,0) \\ (|x-1|-1)^2+b & x \in [0,\infty) \end{bmatrix}$$

g(f(x)) is continuous

at $x = -a$ & at $x = 0$

$1 = b + 1$ & $(a-1)^2 + b = b$

$b = 0$ & $a = 1$

$\Rightarrow a + b = 1$

6. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a}, \frac{1}{b}, 6$ be in A.P., where $a, b > 0$. Then $72(a + b)$ is equal to ________.

Official Ans by NTA (14)

Sol. $a^2 = \frac{b}{16} \Rightarrow \frac{1}{b} = \frac{1}{16a^2}$

$\frac{2}{b} = \frac{1}{a} + 6$

$\frac{1}{8a^2} = \frac{1}{a} + 6$

$\frac{1}{a^2} - \frac{8}{a} - 48 = 0$

$\frac{1}{a} = 12, -4 \quad \Rightarrow \quad a = \frac{1}{12}, -\frac{1}{4}$

$a = \frac{1}{12}, \; a > 0$

$b = 16a^2 = \frac{1}{9}$

$\Rightarrow \; 72(a + b) = 6 + 8 = 14$

7. In ΔABC, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of ΔABC is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of ΔABC, then the value of $2R + r$ (in cm) is equal to ________.

Official Ans by NTA (15)

Sol. $\Delta = \frac{1}{2}.5.12.\sin A = 30$

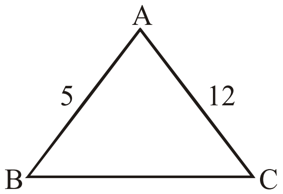
$\sin A = 1$

$A = 90° \Rightarrow BC = 13$

$BC = 2R = 13$

$r = \frac{\Delta}{S} = \frac{30}{15} = 2$

$2R + r = 15$



8. Let n be a positive integer. Let

$$A = \sum_{k=0}^{n} (-1)^k \, {}^nC_k \left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to ________.

Official Ans by NTA (6)

Sol. $A = \sum_{k=0}^{n} {}^nC_k \left[\left(-\frac{1}{2}\right)^k + \left(\frac{-3}{4}\right)^k + \left(\frac{-7}{8}\right)^k + \left(\frac{-15}{16}\right)^k + \left(\frac{-37}{32}\right)^k\right]$

$A = \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \left(1-\frac{31}{32}\right)^n$

$A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \frac{1}{32^n}$

$$A = \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^5}{1-\frac{1}{2^n}}\right) \Rightarrow A = \frac{\left(1-\frac{1}{2^{5n}}\right)}{(2^n-1)}$$

$(2^n - 1)A = 1 - \frac{1}{2^{5n}}$, Given $63A = 1 - \frac{1}{2^{30}}$

Clearly $5n = 30$

$n = 6$

9. Let $\vec{c}$ be a vector perpendicular to the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

If $\vec{c}.(\hat{i} + \hat{j} + 3\hat{k}) = 8$ then the value of $\vec{c}.(\vec{a} \times \vec{b})$ is equal to ________.

Official Ans by NTA (28)

Sol. $\vec{c} = \lambda(\vec{a} \times \vec{b})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$(\vec{a} \times \vec{b}) = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + \hat{k}) . (\hat{i} + \hat{j} + 3\hat{k})$

$\Rightarrow \lambda(4) = 8 \Rightarrow \lambda = 2$

$\vec{c} = 2(\vec{a} \times \vec{b})$

$\vec{c}.(\vec{a} \times \vec{b}) = 2|\vec{a} \times \vec{b}|^2 = 28$

10. Let

$$S_n(x) = \log_{a^{1/2}} x + \log_{a^{1/3}} x + \log_{a^{1/6}} x$$

$$+ \log_{a^{1/11}} x + \log_{a^{1/18}} x + \log_{a^{1/27}} x + ...$$

up to n-terms, where a > 1. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, then value of a is equal to _______.

Official Ans by NTA (16)

Sol. $S_n(x) = (2+3+6+11+18+27+.......+\text{n-terms})\log_a x$

Let $S_1 = 2 + 3 + 6 + 11 + 18 + 27 + ... + T_n$

$S_1 = 2 + 3 + 6 + ... + T_n$

$T_n = 2 + 1 + 3 + 5 + ... + \text{n terms}$

$T_n = 2 + (n - 1)^2$

$S_1 = \Sigma T_n = 2n + \dfrac{(n-1)n(2n-1)}{6}$

$\Rightarrow S_n(x) = \left(2n + \dfrac{n(n-1)(2n-1)}{6}\right)\log_a x$

$S_{24}(x) = 1093$ (Given)

$\log_a x \left(48 + \dfrac{23.24.47}{6}\right) = 1093$

$\log_a x = \dfrac{1}{4}$ .... (1)

$S_{12}(2x) = 265$

$S_{12}(2x) = 265$

$\log_a(2x)\left(24 + \dfrac{11.12.23}{6}\right) = 265$

$\log_a 2x = \dfrac{1}{2}$ .... (2)

(2) – (1)

$\log_a 2x - \log_a x = \dfrac{1}{4}$

$\log_a 2 = \dfrac{1}{4} \Rightarrow a = 16$