FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Friday 29th July, 2022) **TIME : 3 : 00 PM to 06 : 00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. If $z \neq 0$ be a complex number such that $|z - \frac{1}{z}| = 2$, then the maximum value of |z| is:

(A) $\sqrt{2}$ (B) 1

(C) $\sqrt{2} - 1$ (D) $\sqrt{2} + 1$

Official Ans. by NTA (D)

Ans. (D)

Sol. |z – 1/z| = 2

$\left| |z| - \frac{1}{|z|} \right| \leq \left| z - \frac{1}{z} \right| \leq |z| + \frac{1}{|z|}$ Let |z| = r

$\left| r - \frac{1}{r} \right| \leq 2 \leq r + \frac{1}{r}$

$\left| r - \frac{1}{r} \right| \leq 2 \ \& \ r + \frac{1}{r} \geq 2$ always true

$r - \frac{1}{r} \geq -2 \ \& \ r - \frac{1}{r} \leq 2$

$r^2 - 1 \leq 2r$

$r^2 - 2r \leq 1$

$(r - 1)^2 \leq 2$

$r - 1 \leq \sqrt{2}$

$\therefore |z|_{max} = 1 + \sqrt{2}$

2. Which of the following matrices can NOT be obtained from the matrix $\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ by a single elementary row operation?

(A) $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 2 \\ -2 & 7 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$

Official Ans. by NTA (C)

Ans. (C)

Sol. $A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

(1) $R_1 \to R_1 + R_2$; $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ possible

(2) $R_1 \leftrightarrow R_2$; $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ possible

(3) Option is not possible

(4) $R_2 \to R_2 + 2R_1$; $\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$ possible

3. If the system of equations

$x + y + z = 6$

$2x + 5y + \alpha z = \beta$

$x + 2y + 3z = 14$

has infinitely many solutions, then $\alpha + \beta$ is equal to :

(A) 8 (B) 36

(C) 44 (D) 48

Official Ans. by NTA (C)

Ans. (C)

Sol. $x + y + z = 6$ ___(1)

$2x + 5y + \alpha z = \beta$ ___(2)

$x + 2y + 3z = 14$ ____(3)

$x + y = 6 - z$

$x + 2y = 14 - 3z$

On solving

$x = z - 2 \Rightarrow y = 8 - 2z$ in (2)

$2(z - 2) + 5(8 - 2z) + \alpha z = \beta$

$(\alpha - 8) z = \beta - 36$ For having infinite solutions

$\alpha - 8 = 0$ & $\beta - 36 = 0$

$\alpha = 8, \beta = 36$ $(\alpha + \beta = 44)$

4. Let the function

$$f(x)=\begin{cases}\dfrac{\log_e(1+5x)-\log_e(1+\alpha x)}{x} & ;\text{if } x\neq 0\\ 10 & ;\text{if } x=0\end{cases}$$

be continuous at x = 0.

The α is equal to :

(A) 10 (B) –10

(C) 5 (D) –5

Official Ans. by NTA (D)

Ans. (D)

Sol. $$f(x)=\begin{cases}\dfrac{\ln(1+5x)-\ln(1+\alpha x)}{x} & ;x\neq 0\\ 10 & ;x=0\end{cases}$$

$$\lim_{x\to 0}\frac{\ln(1+5x)-\ln(1+\alpha x)}{x}=10$$

Using expension

$$\lim_{x\to 0}\frac{(5x+......)-(\alpha x+......)}{x}=10$$

$5-\alpha=10\Rightarrow \alpha=-5$

5. If [t] denotes the greatest integer $\le$ t, then the value of $\int_0^1\left[2x-|3x^2-5x+2|+1\right]dx$ is:

(A) $\dfrac{\sqrt{37}+\sqrt{13}-4}{6}$ (B) $\dfrac{\sqrt{37}-\sqrt{13}-4}{6}$

(C) $\dfrac{-\sqrt{37}-\sqrt{13}+4}{6}$ (D) $\dfrac{-\sqrt{37}+\sqrt{13}+4}{6}$

Official Ans. by NTA (A)

Ans. (A)

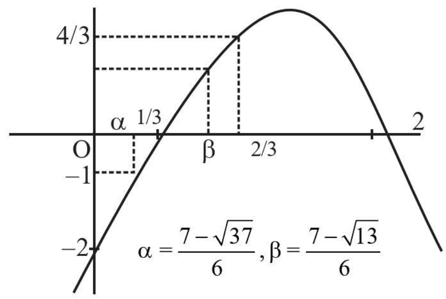
Sol. $$I=\int_0^1\left[2x-|3x^2-3x-2x+2|+1\right]dx$$

$$I=\int_0^1\left[2x-|(3x-2)(x-1)|\right]dx+\int_0^1 1\,dx$$

$$I=\int_0^{2/3}\left[\left(2x-(3x^2-5x+2)\right)\right]dx+\int_{2/3}^1\left(2x+(3x^2-5x+2)\right)dx+1$$

$$I=\int_0^{2/3}\left[-3x^2+7x-2\right]dx+\int_{2/3}^1\left(3x^2-3x+2\right)dx+1$$

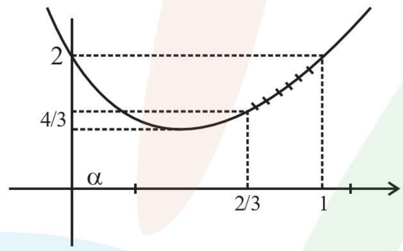
$y=-3x^2+7x-2$



$$\int_0^{\alpha}(-2)dx+\int_{\alpha}^{1/3}(-1)dx+\int_{1/3}^{\beta}0\,dx+\int_{\beta}^{2/3}1.dx$$

$$=-2\alpha-\left(\frac{1}{3}-\alpha\right)+\frac{2}{3}-\beta=-\alpha-\beta+\frac{1}{3}$$

$y=3x^2-3x+2$



When $x\in\left(\dfrac{2}{3},1\right)$

$$3x^2-3x+2\in\left(\frac{4}{3},2\right)$$

$[3x^2-3x+2]=1$

$$\therefore\int_{2/3}^1\left[3x^2-3x+2\right]dx=1\left(1-\frac{2}{3}\right)=\frac{1}{3}$$

Hence $I=\left(\dfrac{1}{3}-(\alpha+\beta)\right)+\left(\dfrac{1}{3}\right)+1$

$$=\frac{5}{3}-\left(\frac{7-\sqrt{37}}{6}+\frac{7-\sqrt{13}}{6}\right)$$

$$=\frac{-2}{3}+\frac{\sqrt{37}+\sqrt{13}}{6}$$

$$=\frac{\sqrt{37}+\sqrt{13}-4}{6}$$

6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and $a_{n+2} = 3a_{n+1} - 2a_n + 1, \forall n \geq 0$.

Then $a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}$ is equal to:

(A) 483 (B) 528 (C) 575 (D) 624

Official Ans. by NTA (B)

Ans. (B)

Sol. $a_0 = 0, a_1 = 0$

$a_{n+2} = 3a_{n+1} - 2a_{n+1} : n \geq 0$

$a_{n+2} - a_{n+1} = 2(a_{n+1} - a_n) + 1$

n = 0 $\quad a_2 - a_1 = 2(a_1 - a_0) + 1$

n = 1 $\quad a_3 - a_2 = 2(a_2 - a_1) + 1$

n = 2 $\quad a_4 - a_3 = 2(a_3 - a_2) + 1$

n = n $\quad a_{n+2} - a_{n+1} = 2(a_{n+1} - a_n) + 1$

$(a_{n+2} - a_1) - 2(a_{n+1} - a_0) - (n+1) = 0$

$a_{n+2} = 2a_{n+1} + (n+1)$

$n \rightarrow n-2$

$a_n - 2a_{n-1} = n - 1$

Now $a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}$

$= (a_{25} - 2a_{24})(a_{23} - 2a_{22}) = (24)(22) = 528$

7. $\sum_{r=1}^{20}(r^2+1)(r!)$ is equal to:

(A) 22! – 21! (B) 22! – 2 (21!)

(C) 21! – 2 (20!) (D) 21! – 20!

Official Ans. by NTA (B)

Ans. (B)

Sol. $\sum_{x=1}^{20}(r^2+1)r!$

$\sum_{x=1}^{20}\left((r+1)^2 - 2r\right)r!$

$\sum_{x=1}^{20}\left((r+1)(r+1)! - r.r!\right) - \sum_{r=1}^{20} r.r!$

$\sum_{x=1}^{20}\left((r+1)(r+1)! - r.r!\right) - \sum_{r=1}^{20}\left((r+1)! - r!\right)$

$= (21.\lfloor 21 - 1) - (\lfloor 21 - 1)$

$= 20.21! = 22! - 2.21!$

8. For $I(x) = \int \frac{\sec^2 x - 2022}{\sin^{2022} x}dx$, if $I\left(\frac{\pi}{4}\right) = 2^{1011}$, then

(A) $3^{1010} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

(B) $3^{1010} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

(C) $3^{1011} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

(D) $3^{1011} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

Official Ans. by NTA (A)

Ans. (A)

Sol. $I(x) = \int \underset{II}{\sec^2 x}.\underset{I}{\sin^{-2022} x}\, dx - 2022\int \sin^{-2022} x\, dx$

$= \tan x.(\sin x)^{-2022} + \int (2022)\tan x.(\sin x)^{-2023}\cos x dx$

$-2022\int (\sin x)^{-2022} dx$

$I(x) = (\tan x)(\sin x)^{-2022} + C$

At $X = \pi/4$, $2^{1011} = \left(\frac{1}{\sqrt{2}}\right)^{-2022} + C \therefore C = 0$

Hence $I(x) = \frac{\tan x}{(\sin x)^{2022}}$

$I(\pi/6) = \frac{1}{\sqrt{3}\left(\frac{1}{2}\right)^{2022}} = \frac{2^{2022}}{\sqrt{3}}$

$I(\pi/3) = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}} = \frac{2^{2022}}{(\sqrt{3})^{2021}} = \frac{1}{3^{1010}} I\left(\frac{\pi}{6}\right)$

$3^{1010} I(\pi/3) = I(\pi/6)$

9. If the solution curve of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passes through the point (2,1) and (k + 1,2), k > 0, then

(A) $2\tan^{-1}\left(\frac{1}{k}\right) = \log_e(k^2+1)$

(B) $\tan^{-1}\left(\frac{1}{k}\right) = \log_e(k^2+1)$

(C) $2\tan^{-1}\left(\frac{1}{k+1}\right) = \log_e(k^2+2k+2)$

(D) $2\tan^{-1}\left(\frac{1}{k}\right) = \log_e\left(\frac{k^2+1}{k^2}\right)$

Official Ans. by NTA (A)

Ans. (A)

Sol. $\frac{dy}{dx}=\frac{x+y-2}{x-y}=\frac{(x-1)+(y-1)}{(x-1)-(y-1)}$

$x-1=X,\ y-1=Y$

$\frac{dY}{dX}=\frac{X+Y}{X-Y}$

$Y=VX \qquad \frac{dY}{dX}=V+X\frac{dV}{dX}$

$V+X\frac{dV}{dX}=\frac{1+V}{1-V} \qquad X\frac{dV}{dX}=\frac{V^2+1}{1-V}$

$\int\frac{1-V}{1+V^2}dV=\int\frac{dX}{X}$

$\int\frac{dV}{1+V^2}-\frac{1}{2}\int\frac{2VdV}{1+V^2}=\int\frac{dX}{X}$

$\tan^{-1}V-\frac{1}{2}\ln(1+V^2)=\ln X+c$

$\tan^{-1}\left(\frac{Y}{X}\right)-\frac{1}{2}\ln\left(1+\frac{Y^2}{X^2}\right)=\ln(X)+c$

$\tan^{-1}\left(\frac{y-1}{x-1}\right)-\frac{1}{2}\ln\left(1+\frac{(y-1)^2}{(x-1)^2}\right)=\ln(x-1)+c$

Passes through (2,1)

$0-\frac{1}{2}\ \ln 1=\ln 1+c\ \therefore\ c=0$

Passes through (k + 1, 2)

$\therefore\ \tan^{-1}\left(\frac{1}{k}\right)-\frac{1}{2}\ln\left(1+\frac{1}{k^2}\right)=\ln k$

$2\tan^{-1}\left(\frac{1}{k}\right)=\ln\left(\frac{1+k^2}{k^2}\right)+2\ln k$

$2\tan^{-1}\left(\frac{1}{k}\right)=\ln(1+k^2)$

10. Let y = y (x) be the solution curve of the differential equation $\frac{dy}{dx}+\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)$ $y=\frac{(x+3)}{x+1}, x>-1$, which passes through the point (0,1). Then y (1) is equal to:

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $\frac{5}{2}$ (D) $\frac{7}{2}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $\frac{dy}{dx}+\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y=\frac{x+3}{x+1}$

$\int p(x)dx \qquad\qquad \text{I.F.}=e^{\int p(x)dx}$

$\int p(x)dx=\int\frac{(2x^2+11x+13)dx}{(x+1)(x+2)(x+3)}$

Using partial fraction

$\frac{2x^2+11x+13}{(x+1)(x+2)(x+3)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}$

$A=\frac{4}{2}=2$

$B=1$

$C=-1$

$\because \int p(x)dx=A\ln(x+1)+B\ln(x+2)+c\ln(x+3)$

$=\ln\left(\frac{(x+1)^2(x+2)}{x+3}\right)$

$\text{I.F.}=e^{\int p(x)dx}=\frac{(x+1)^2(x+2)}{(x+3)}$

Solution $y(\text{IF})=\int Q.(\text{IF})dx$

$y\left(\frac{(x+1)^2(x+2)}{x+3}\right)=\int\left(\frac{x+3}{x+1}\right)\frac{(x+1)^{2}(x+2)}{(x+3)}dx$

$y\left(\frac{(x+1)^2(x+2)}{x+3}\right)=\frac{x^3}{3}+\frac{3x^2}{2}+2x+c$

Passes through (0, 1) $C=\frac{2}{3}$

Now put x = 1

$\Rightarrow y(1)=\frac{3}{2}$

11. Let m_1, m_2 be the slopes of two adjacent sides of a square of side a such that $a^2+11a+3(m_2^2+m_2^2)=220$. If one vertex of the square is $(10(\cos\alpha-\sin\alpha), 10(\sin\alpha+\cos\alpha))$, where $\alpha\in\left(0,\frac{\pi}{2}\right)$ and the equation of one diagonal is $(\cos\alpha-\sin\alpha)x+(\sin\alpha+\cos\alpha)y=10$, then $72(\sin^4\alpha+\cos^4\alpha)+a^2-3a+13$ is equal to:

(A) 119 (B) 128

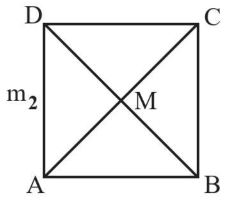
(C) 145 (D) 155

Official Ans. by NTA (B)

Ans. (B)

Sol. $m_1m_2 = -1$

$a^2 + 11a + 3\left(m_1^2 + \frac{1}{m_1^2}\right) = 220$



Eq. of AC

AC = (cosα – sinα) + (sinα + cosα) y = 10

BD = (sinα – cosα) x + (sinα – cosα) y = 0

(10 (cosα – sinα), 10 (sinα – cosα))

Slope of AC = $\left(\frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}\right) = \tan\theta = M$

Eq. of line making an angle π_4 with AC

$$m_1, m_2 = \frac{m \pm \tan\frac{\pi}{4}}{1 \pm m\tan\frac{\pi}{4}}$$

$$= \frac{m+1}{1-m} \text{ or } \frac{m-1}{1+m}$$

$$\frac{\frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} + 1}{1 - \left(\frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}\right)}, \frac{\frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} - 1}{1 + \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}}$$

m_1, m_2 = tanα , cotα

mid point of AC & BD

= M (5(cosα–sinα), 5(cosα+sinα))

B (10(cosα–sinα), 10(cosα+sinα))

$a = AB = \sqrt{2}\,BM = \sqrt{2}\left(5\sqrt{2}\right) = 10$

a = 10

$\because a^2 + 11a + 3\left(m_1^2 + \frac{1}{m_1 2}\right) = 220$

100 + 110 + 3 ($\tan^2\alpha + \cot^2\alpha$) = 220

Hence $\tan^2\alpha = 3$, $\tan^2\alpha = \frac{1}{3} \Rightarrow \alpha = \frac{\pi}{3} \text{ or } \frac{\pi}{6}$

Now 72 ($\sin^4\alpha + \cos^4\alpha$) + a^2 – 3a + 13

$= 72\left(\frac{9}{16} + \frac{1}{16}\right) + 100 - 30 + 13$

$= 72\left(\frac{5}{8}\right) + 83 = 45 + 83 = 128$

12. The number of elements in the set $S = \left\{x \in \mathbb{R} : 2\cos\left(\frac{x^2 + x}{6}\right) = 4^x + 4^{-x}\right\}$ is:

(A) 1 (B) 3

(C) 0 (D) infinite

Official Ans. by NTA (A)

Ans. (A)

Sol. $2\cos\left(\frac{x^2 + x}{6}\right) = 4^x + 4^{-x}$

L.H.S ≤ 2. & R.H.S. ≥ 2

Hence L.H.S = 2 & R.H.S = 2

$2\cos\left(\frac{x^2 + x}{6}\right) = 2 \quad 4^x + 4^{-x} = 2$

Check x = 0 Possible hence only one solution.

13. Let A (α, –2), B (α, 6) and $C\left(\frac{\alpha}{4}, -2\right)$ be vertices of a ΔABC. If $\left(5, \frac{\alpha}{4}\right)$ is the circumcentre of ΔABC, then which of the following is NOT correct about ΔABC:

(A) ares is 24 (B) perimeter is 25

(C) circumradius is 5 (D) inradius is 2

Official Ans. by NTA (B)

Ans. (B)

Sol. A (α, –2) : B (α, 6) : $C\left(\frac{\alpha}{4}, -2\right)$

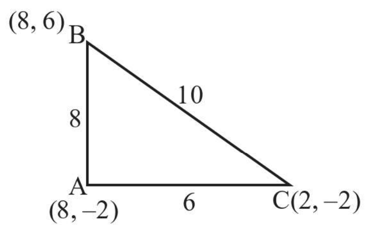
since AC is perpendicular to AB.

So, ΔABC is right angled at A.

Circumcentre = mid point of BC. = $\left(\frac{5\alpha}{8}, 2\right)$

$\therefore \frac{5\alpha}{8} = 5 \ \& \ \frac{\alpha}{4} = 2$

α = 8



Area = $\frac{1}{2}(6)(8) = 24$

Perimeter = 24

Circumradius = 5

Inradius = $\frac{\Delta}{s} = \frac{24}{12} = 2$

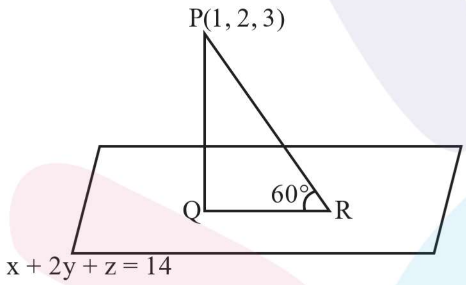
14. Let Q be the foot of perpendicular drawn from the point P (1, 2, 3) to the plane x + 2y + z = 14. If R is a point on the plane such that ∠PRQ = 60°, then the area of ΔPQR is equal to:

(A) $\frac{\sqrt{3}}{2}$ (B) $\sqrt{3}$

(C) $2\sqrt{3}$ (D) 3

Official Ans. by NTA (B)

Ans. (B)



Sol. Length of perpendicular

$$PQ = \left|\frac{1+4+3-14}{\sqrt{6}}\right| = \sqrt{6}$$

QR = (PQ) cot 60° = $\sqrt{2}$

∴ Area of ΔPQR = $\frac{1}{2}(PQ)(QR) = \sqrt{3}$

15. If (2, 3, 9), (5, 2, 1), (1, λ, 8) and (λ, 2, 3) are coplanar, then the product of all possible values of λ is:

(A) $\frac{21}{2}$ (B) $\frac{59}{8}$

(C) $\frac{57}{8}$ (D) $\frac{95}{8}$

Official Ans. by NTA (D)

Ans. (D)

Sol. A(2, 3, 9) ; B(5, 2, 1) ; C(1, λ, 8) ; D(λ, 2, 3)

$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = 0$$

$$\begin{vmatrix} 3 & -1 & -8 \\ -1 & \lambda-3 & -1 \\ \lambda-2 & -1 & -6 \end{vmatrix} = 0$$

⇒ [–6(λ – 3) –1] –8(1 –(λ – 3) (λ – 2)) + (6 + (λ – 2) = 0

$3(-6\lambda + 17) - 8(-\lambda^2 + 5\lambda - 5) + (\lambda + 4) = 8$

$8\lambda^2 - 57\lambda + 95 = 0$

$\lambda_1 \lambda_2 = \frac{95}{8}$

16. Bag I contains 3 red, 4 black and 3 white balls and Bag II contains 2 red, 5 black and 2 white balls. One ball is transferred from Bag I to Bag II and then a ball is draw from Bag II. The ball so drawn is found to be black in colour. Then the probability, that the transferred ball is red, is:

(A) $\frac{4}{9}$ (B) $\frac{5}{18}$ (C) $\frac{1}{6}$ (D) $\frac{3}{10}$

Official Ans. by NTA (B)

Ans. (B)

Sol.

3R	2R
4B	5B
3W	2W

A : Drown ball from boy II is black

B : Red ball transferred

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{3}{9} \times \frac{5}{10}}{\frac{3}{9} \times \frac{5}{10} + \frac{4}{9} \times \frac{6}{10} + \frac{3}{9} \times \frac{5}{10}}$$

$$= \frac{15}{15+24+15} = \frac{15}{54} = \frac{5}{18}$$

17. Let S = {z = x + iy : |z – 1 + i| ≥ | z |, |z| < 2, |z + i| = |z –1|}. Then the set of all values of x, for which w = 2x + iy ∈ S for some y ∈ $\mathbb{R}$, is

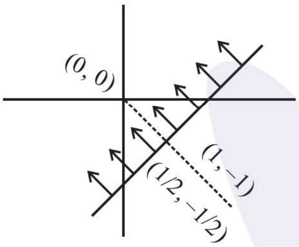
(A) $\left(-\sqrt{2}, \frac{1}{2\sqrt{2}}\right]$ (B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$

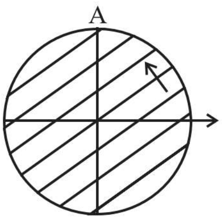
(C) $\left(-\sqrt{2}, \frac{1}{2}\right]$ (D) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$

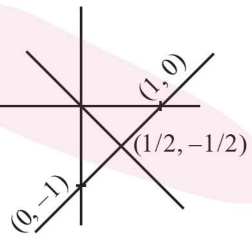
Official Ans. by NTA (B)

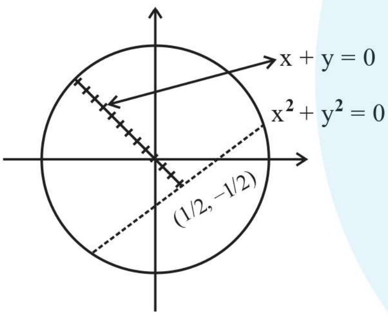
Ans. (B)

Sol. $|z - 1 + i| \geq |z|$; $|z| < 2$; $|z + i| = |z - 1|$









Hence

w = 2x + iy ∈ S

$2x \leq \frac{1}{2} \quad \therefore x \leq \frac{1}{4}$

Now

$(2x)^2 + (2x)^2 < 4$

$x^2 < \frac{1}{2} \Rightarrow x \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\therefore x \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{4}\right]$

18. Let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar concurrent vectors such that angles between any two of them is same. If the product of their magnitudes is 14 and $(\vec{a}\times\vec{b}).(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{c}).(\vec{c}\times\vec{a}) + (\vec{c}\times\vec{a}).(\vec{a}\times\vec{b}) = 168$ then $|\vec{a}| + |\vec{b}| + |\vec{c}|$ is equal to:

(A) 10 (B) 14

(C) 16 (D) 18

Official Ans. by NTA (C)

Ans. (C)

Sol. $|\vec{a}||\vec{b}||\vec{c}| = 14$

$\vec{a} \wedge \vec{b} = \vec{b} \wedge \vec{c} = \vec{c} \wedge \vec{a} = \theta = \frac{2\pi}{3}$

So, $\vec{a}.\vec{b} = -\frac{1}{2}ab,\ \vec{b}.\vec{c} = -\frac{1}{2}bc,\ \vec{a}.\vec{c}. = -\frac{1}{2}ac$

(let)

$(\vec{a}\times\vec{b}).(\vec{b}\times\vec{c}) = (\vec{a}.\vec{b})(\vec{b}.\vec{c}) - (\vec{a}.\vec{c})(\vec{b}.\vec{b})$

$= \frac{1}{4}ab^2c + \frac{1}{2}ab^2c = \frac{3}{4}ab^2c$

Similarly

$(\vec{b}\times\vec{c}).(\vec{c}\times\vec{a}) = \frac{3}{4}abc^2$

$(\vec{c}\times\vec{a}).(\vec{a}\times\vec{b}) = \frac{3}{4}a^2bc$

$168 = \frac{3}{4}abc(a + b + c)$

So, (a + b + c) = 16

19. The domain of the function $f(x) = \sin^{-1}\left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7}\right)$ is :

(A) [1, ∞) (B) (–1, 2]

(C) [–1, ∞) (D) (–∞,2]

Official Ans. by NTA (C)

Ans. (C)

Sol. $f(x) = \sin^{-1}\left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7}\right)$ Domain

$\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \geq -1$ and $\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \leq 1$

$2x^2 - x + 9 \geq 0$ and $5x \geq -5 \Rightarrow x \geq -1$

x ∈ R

Hence Domain x ∈ [–1, ∞)

20. The statement $(p \Rightarrow q) \vee (p \Rightarrow r)$ is NOT equivalent to:

(A) $(p \wedge (\sim r)) \Rightarrow q$ (B) $(\sim q) \Rightarrow ((\sim r) \vee p)$

(C) $p \Rightarrow (q \vee r)$ (D) $(p \wedge (\sim q)) \Rightarrow r$

Official Ans. by NTA (B)

Ans. (B)

Sol. $(p \to q) \vee (p \to r)$

$(\sim p \vee q) \vee (\sim p \vee r)$

$= \sim p \vee (q \vee r)$

$= p \to (q \vee r) \equiv (3)$ is true.

Now (1) $(p \wedge \sim r) \to q$

$\sim(p \wedge \sim r) \vee q = (\sim p \vee r) \vee q$

$= \sim p \vee (r \vee q) = p \to (q \vee r)$

(4) $(p \wedge \sim q) \to r = p \to (q \vee r)$

SECTION-B

1. The sum and product of the mean and variance of a binomial distribution are 82.5 and 1350 respectively. They the number of trials in the binomial distribution is:

Official Ans. by NTA (96)

Ans. (96)

Sol. Let, mean = m = np

& variance = v = npq, p + q = 1

Sum $= m + v = \frac{165}{2}$

Product = mv = 1350

On solving,

$m = np = 60$ & $v = npq = \frac{45}{2} \therefore q = \frac{3}{8} \therefore P = \frac{5}{8}$

Hence n = 96

2. Let α, β (α > β) be the roots of the quadratic equation $x^2 - x - 4 = 0$. If $P_n = \alpha^n - \beta^n$, $n \in \mathbb{N}$, then

$\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$ is equal to ______.

Official Ans. by NTA (16)

Ans. (16)

Sol. $P_n = \alpha^n - \beta^n$ $\qquad x^2 - x - 4 = 0$

$$\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}} \quad \_\_(1)$$

As $P_n - P_{n-1} = (\alpha^n - \beta^n) - (\alpha^{n-1} - \beta^{n-1})$

$= \alpha^{n-2}(\alpha^2 - \alpha) - \beta^{n-2}(\beta^2 - \beta)$

$= 4(\alpha^{n-2} - \beta^{n-2})$

$P_n - P_{n-1} = 4P_{n-2}$

Hence Expression (1)

$$\frac{P_{16}(P_{15} - P_{14}) - P_{15}(P_{15} - P_{14})}{P_{13}P_{14}}$$

$$= \frac{(P_{15} - P_{14})(P_{16} - P_{15})}{P_{13}P_{14}} = \frac{(4P_{13})(4P_{14})}{P_{13}P_{14}} = 16$$

3. Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}$. For $k \in \mathbb{N}$, if $X' A^k X = 33$, then k is equal to:

Official Ans. by NTA (10)

Ans. (Dropped or 10)

Sol. $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}$

$X^T A^K X = 33$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 33$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 33$$

$$\text{As } A^2 = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 1 & 0 & 24 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{10}=\begin{bmatrix}1&0&6\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&24\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&30\\0&1&0\\0&0&1\end{bmatrix}$$

for $K \to$ Even $A^K=\begin{bmatrix}1&0&3K\\0&1&0\\0&0&1\end{bmatrix}$

$X^TA^KX = 33$ (This is not correct)

$$[1\ \ 1\ \ 1]\begin{bmatrix}1&0&3K\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$=[1\ \ 1\ \ 3K+1]\begin{bmatrix}1\\1\\1\end{bmatrix}=[3K+3]$$

$\therefore\ 3K + 3 = 33\ \therefore\ K = 10$

But it should be dropped as 33 is not matrix

If K is odd

$X^TA^KX = 33$

$X^TAA^{K-1}X = 33$

$$\begin{bmatrix}1&1&1\end{bmatrix}\begin{bmatrix}-1&2&3\\0&1&6\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&3k-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=33$$

$$\begin{bmatrix}-1&3&8\end{bmatrix}\begin{bmatrix}3k-2\\1\\1\end{bmatrix}=[33]$$

$[-3k + 13] = [33]$

$k = 20/3$ (not possible)

4. The number of natural numbers lying between 1012 and 23421 that can be formed using the digits 2, 3, 4, 5, 6 (repetition of digits is not allowed) and divisible by 55 is____,

Official Ans. by NTA (6)

Ans. (6) Sol. 4

digit numbers For

divisibility by 55, no. should be

div. by 5 and 11 both

			5
a	b	c	d

Also, for divisibility by 11

$a + c = b + 5$

for b = 1 $\quad a = 2,\ c = 4$

$\qquad\qquad a = 4,\ c = 2$

for b = 2 $\quad a = 3,\ c = 4$

$\qquad\qquad a = 4,\ c = 3$

for b = 3 $\quad a = 6,\ c = 2$

$\qquad\qquad a = 2,\ c = 6$

$\therefore$ 6 possible four digit no.s are div. by 55

(II) 5 digit number is not possible

(Not possible)

5. If $\sum_{k=1}^{10} K^2\left(10_{C_K}\right)^2 = 22000L$, then L is equal to ___.

Official Ans. by NTA (221)

Ans. (221)

Sol. $\sum_{K=1}^{10} K^2\left({}^{10}C_K\right)^2$

$$\sum_{K=1}^{10}\left(K.{}^{10}C_K\right)^2=\sum_{K=1}^{10}\left(10.{}^{9}C_{K-1}\right)^2$$

$$=100\sum_{K=1}^{10}{}^9C_{K-1}.{}^9C_{10-K}$$

$$=100\left({}^{18}C_9\right)=100\left(\frac{18!}{9!9!}\right)$$

$\Rightarrow 4862000 = 22000L$

Hence L = 221

6. If [t] denotes the greatest integer $\le$ t, then number of points, at which the function f(x) = 4 | 2x + 3| + $9\left[x+\frac{1}{2}\right]-12\ [x + 20]$ is not differentiable in the open interval (–20, 20), is____.

Official Ans. by NTA (79)

Ans. (79)

Sol. $f(x) = 4|2x + 3| + 9\left[x+\frac{1}{2}\right]-12[x + 20]$

$x \in (-20, 20)$

f(x) is not Diff. at $x = I \in \{-19, -18, ...0,...19\} = 39$

at $x = I + \frac{1}{2}$, f(x) Non diff. at 39 points

Check at $x = \frac{-3}{2}$ Discount at $x = \frac{-3}{2}$ $\therefore$ N. R(1)

No. of point of non-differentiabilty

= 39 + 39 + 1 = 79

7. If the tangent to the curve $y = x^3 - x^2 + x$ at the point (a , b) is also tangent to the curve $y = 5x^2 + 2x - 25$ at the point (2, –1), then $|2a + 9b|$ is equal to ______.

Official Ans. by NTA (195)

Ans. (195)

Sol. $y = 5x^2 + 2x - 25$ $\quad$ P(2, –1)

$y' = 10x + 2$

$y'_P = 22$

$\therefore$ tangent to curve at P

$y + 1 = 22 (x - 2)$

$y = 22x - 45$

$y = x^3 - x^2 + x$

$\left.\frac{dy}{dx}\right|_{C_2} = 3x^2 - 2x + 1$

$\left.\frac{dy}{dx}\right|_{Q} = 3a^2 - 2a + 1$

Hence $3a^2 - 2a + 1 = 22$

$\therefore\ 3a^2 - 2a - 21 = 0$

$3a^2 - 9a + 7a - 21 = 0$

$(3a + 7)(a - 3) = 0$ $\quad a = 3$, $a = -7/3$

from curve $b = a^3 - a^2 + a$

$a = 3$

$b = 21$ $\quad |2a + 9b| = 195$

at $a = -7/3$ tangent will be parallel

Hence it is rejected

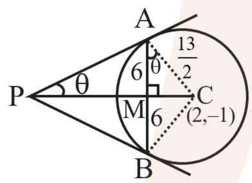
8. Let AB be a chord of length 12 of the circle $(x - 2)^2 + (y + 1)^2 = \frac{169}{4}$.

If tangents drawn to the circle at points A and B intersect at the point P, then five times the distance of point P from chord AB is equal to ___.

Official Ans. by NTA (72)

Ans. (72)

Sol.



$\cos\theta = \frac{6}{\frac{13}{2}} = \frac{12}{13}$

$\sin\theta = \frac{5}{13}$

$PM = AM \cot\theta$

$PM = 6\left(\frac{12}{5}\right)$ $\quad \therefore\ 5(PM) = 72$

9. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2, \vec{a}.\vec{b} = 3$ and $|\vec{a} \times \vec{b}|^2 = 75$. Then $|\vec{a}|^2$ is equal to_____.

Official Ans. by NTA (14)

Ans. (14)

Sol. $\left|\vec{a} + \vec{b}\right|^2 = \left|\vec{a}\right|^2 + 2\left|\vec{b}\right|^2$; $\vec{a}.\vec{b} = 3$

As $\cancel{|\vec{a}|^2} + |\vec{b}|^2 + 2\vec{a}.\vec{b} = \cancel{|\vec{a}|^2} + 2|\vec{b}|^2$

$|\vec{b}|^2 = 2\vec{a}.\vec{b} = 6$

$|\vec{a} \times \vec{b}|^2 = 75$

$|\vec{a}|^2|\vec{b}|^2 - \left(\vec{a}.\vec{b}\right)^2 = 75$

$6|\vec{a}|^2 - 9 = 75 \Rightarrow |\vec{a}|^2 = 14$

10. Let

$$S=\{(x,y)\in\mathbb{N}\times\mathbb{N}:9(x-3)^2+16(y-4)^2\le 144\}$$

and $T=\{(x,y)\in\mathbb{R}\times\mathbb{R}:(x-7)^2+(y-4)^2\le 36\}$.

The $n(S\cap T)$ is equal to______.

Official Ans. by NTA (27)

Ans. (27)

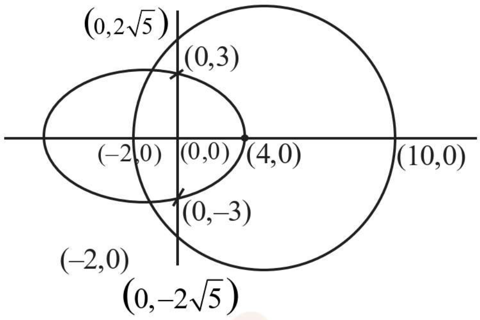
Sol. $S:\dfrac{(x-3)^2}{16}+\dfrac{(y-4)^2}{9}\le 1$; $x, y \in \{1, 2, 3, ...\}$

$T: (x-7)^2+(y-4)^2 \le 36$ $x, y \in R$

Let $x-3 = x : y-4 = y$

$S:\dfrac{x^2}{16}+\dfrac{y^2}{9}\le 1$; $x \in \{-2,-1,0,1, ...\}$

$T: (x-4)^2+y^2\le 36$; $y \in \{-3,-2,-1,0, ...\}$



$S\cap T = (-2, 0), (-1, 0), ...(4, 0) \rightarrow (7)$

$(-1, 1), (0, 1), ...(3, 1) \rightarrow (5)$

$(-1, -1), (0, -1), ...(3, -1) \rightarrow (5)$

$(-1, 2), (0, 2), (1, 2), (2, 2) \rightarrow (4)$

$(-1, -2), (0, -2), (1, -2), (2, -2) \rightarrow (4)$

$(0, 3)\ (0, -3) \rightarrow (2)$