

**FINAL JEE-MAIN EXAMINATION – JUNE, 2022**

(Held On Saturday 25<sup>th</sup> June, 2022)

TIME : 3 : 00 PM to 6 : 00 PM

**MATHEMATICS**

**TEST PAPER WITH SOLUTION**

**SECTION-A**

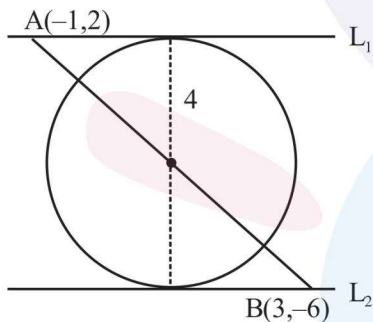
1. Let a circle C touch the lines  $L_1 : 4x - 3y + K_1 = 0$  and  $L_2 : 4x - 3y + K_2 = 0$ ,  $K_1, K_2 \in \mathbb{R}$ . If a line passing through the centre of the circle C intersects  $L_1$  at  $(-1, 2)$  and  $L_2$  at  $(3, -6)$ , then the equation of the circle C is

- (A)  $(x - 1)^2 + (y - 2)^2 = 4$
- (B)  $(x + 1)^2 + (y - 2)^2 = 4$
- (C)  $(x - 1)^2 + (y + 2)^2 = 16$
- (D)  $(x - 1)^2 + (y - 2)^2 = 16$

Official Ans. by NTA (C)

Ans. (C)

Sol.



$L_1 : 4x - 3y + K_1 = 0$

$L_2 : 4x - 3y + K_2 = 0$

now

$-4 - 6 + K_1 = 0 \Rightarrow K_1 = 10$

$12 + 18 + K_2 = 0 \Rightarrow K_2 = -30$

$\Rightarrow$  Tangent to the circle are

$4x - 3y + 10 = 0$

$4x - 3y - 30 = 0$

Length of diameter  $2r = \frac{|10+30|}{5} = 8$

$\Rightarrow r = 4$

Now centre is mid point of A & B

$x = 1, y = -2$

Equation of circle

$(x - 1)^2 + (y + 2)^2 = 16$  Ans.

2. The value of  $\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$  is equal to

- (A)  $\frac{\pi^2}{4}$
- (B)  $\frac{\pi^2}{2}$
- (C)  $\frac{\pi}{4}$
- (D)  $\frac{\pi}{2}$

Official Ans. by NTA (C)

Ans. (C)

Sol.  $\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx \dots(1)$

Use King's property

$I = \int_0^{\pi} \frac{e^{-\cos x} \sin x}{(1 + \cos^2 x)(e^{-\cos x} + e^{\cos x})} dx \dots(2)$

On adding equation (1) and (2), we get

$2I = \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = 2 \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$

On putting  $\cos x = t$ , we get

$I = \int_0^1 \frac{dt}{1+t^2} = (\tan^{-1} t)_0^1 = \frac{\pi}{4}$

3. Let a, b and c be the length of sides of a triangle

ABC such that  $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$ . If r and R

are the radius of incircle and radius of circumcircle of the triangle ABC, respectively,

then the value of  $\frac{R}{r}$  is equal to

- (A)  $\frac{5}{2}$
- (B) 2
- (C)  $\frac{3}{2}$
- (D) 1

Official Ans. by NTA (A)

Ans. (A)

**Sol.**  $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9} = \lambda$

$a + b = 7\lambda, b + c = 8\lambda, a + c = 9\lambda$

$\Rightarrow a + b + c = 12\lambda$

Now  $a = 4\lambda, b = 3\lambda, c = 5\lambda$

$\therefore c^2 = b^2 + a^2$

$\angle C = 90^\circ$

$\Delta = \frac{1}{2}ab \sin C = \frac{1}{2}ab$

$\frac{R}{r} = \frac{c}{2 \sin C} \times \frac{s}{\Delta} = \frac{c}{2} \times \frac{6\lambda}{\frac{1}{2}ab} = \frac{c}{ab} \times 6\lambda = \frac{5}{2}$

4. Let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be a function such that  $f(x+y) = 2f(x)f(y)$  for natural numbers  $x$  and  $y$ . If  $f(1) = 2$ , then the value of  $\alpha$  for which

$$\sum_{k=1}^{10} f(\alpha+k) = \frac{512}{3}(2^{20}-1)$$

holds, is

- (A) 2 (B) 3  
(C) 4 (D) 6

**Official Ans. by NTA (C)**

**Ans. (C)**

**Sol.**  $f : \mathbb{N} \rightarrow \mathbb{R}, f(x+y) = 2f(x)f(y) \dots(1)$

$f(1) = 2,$

$$\sum_{k=1}^{10} f(\alpha+k) = 2f(\alpha) \sum_{k=1}^{10} f(k)$$

$= 2f(\alpha)(f(1)+f(2)+\dots+f(10)) \dots(2)$

From (1)

$f(2) = 2f^2(1) = 2^3$

$f(3) = 2f(2)f(1) = 2^5$

$\vdots$

$f(10) = 2^9 f^{10}(1) = 2^{19}$

$f(\alpha) = 2^{2\alpha-1}; \alpha \in \mathbb{N}$

from (2)

$$\sum_{k=1}^{10} f(\alpha+k) = 2(2^{2\alpha-1})(2+2^3+2^5+\dots+2^{19})$$

$$\frac{512}{3}(2^{20}-1) = 2^{2\alpha} \left( \frac{2(2^{20}-1)}{3} \right)$$

Hence  $\alpha = 4$

5. Let  $A$  be a  $3 \times 3$  real matrix such that

$A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  and  $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$

If  $X = (x_1, x_2, x_3)^T$  and  $I$  is an identity matrix

of order 3, then the system  $(A - 2I)X = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$

has

- (A) no solution  
(B) infinitely many solutions  
(C) unique solution  
(D) exactly two solutions

**Official Ans. by NTA (B)**

**Ans. (B)**

**Sol.**  $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\Rightarrow c_1 = 1, c_2 = 1, c_3 = 2$

$A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + a_1 \\ c_2 + a_2 \\ c_3 + a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow a_1 = -2, a_2 = -1, a_3 = -1$

$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow b_1 = 3, b_2 = 2, b_3 = 1$

$\Rightarrow A = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

$\Rightarrow A - 2I = \begin{bmatrix} -4 & 3 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$|A - 2I| = 0$

Now,  $\begin{bmatrix} -4 & 3 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

$-4x_1 + 3x_2 + x_3 = 4 \dots(1)$

$-x_1 + x_3 = 1 \dots(2)$

$-x_1 + x_2 = 1 \dots(3)$

$(1) - [(2) + 3(3)]$

$0 = 0 \Rightarrow$  infinite solutions

6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = x^3 + x - 5$ . If  $g(x)$  is a function such that  $f(g(x)) = x$ ,  $\forall x \in \mathbb{R}$ , then  $g'(63)$  is equal to \_\_\_\_\_.

- (A)  $\frac{1}{49}$  (B)  $\frac{3}{49}$   
 (C)  $\frac{43}{49}$  (D)  $\frac{91}{49}$

Official Ans. by NTA (A)

Ans. (A)

Sol.  $f(x) = x^3 + x - 5$   
 $\Rightarrow f'(x) = 3x^2 + 1 \Rightarrow$  increasing function  
 $\Rightarrow$  invertible  
 $\Rightarrow g(x)$  is inverse of  $f(x)$   
 $\Rightarrow g(f(x)) = x$   
 $\Rightarrow g'(f(x))f'(x) = 1$   
 $f(x) = 63$   
 $\Rightarrow x^3 + x - 5 = 63$   
 $\Rightarrow x = 4$   
 put  $x = 4$

$$g'(f(4))f'(4) = 1$$

$$g'(63) \times 49 = 1 \quad \{f(4) = 63\}$$

$$g'(63) = \frac{1}{49}$$

7. Consider the following two propositions:

$$P1 : \sim(p \rightarrow \sim q)$$

$$P2 : (p \wedge \sim q) \wedge ((\sim p) \vee q)$$

If the proposition  $p \rightarrow ((\sim p) \vee q)$  is evaluated as FALSE, then:

- (A) P1 is TRUE and P2 is FALSE  
 (B) P1 is FALSE and P2 is TRUE  
 (C) Both P1 and P2 are FALSE  
 (D) Both P1 and P2 are TRUE

Official Ans. by NTA (C)

Ans. (C)

Sol.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$	$p \rightarrow \sim q$	$\sim(p \rightarrow \sim q)$	$p \wedge \sim q$	$p_2$
T	T	F	F	T	T	F	F	F	F
T	F	F	T	F	F	T	T	T	F
F	T	T	F	T	T	F	F	F	F
F	F	T	T	T	T	T	T	F	F

$p \rightarrow (\sim p \vee q)$  is F when p is true q is false  
 From table  
 P1 & P2 both are false

8. If  $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \dots + \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$ , then the remainder when K is divided by 6 is  
 (A) 1 (B) 2  
 (C) 3 (D) 5

Official Ans. by NTA (D)

Ans. (D)

Sol.  $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \frac{1}{2^3 \cdot 3^8} + \dots + \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$

$$K = 2^9 + 2^8 \cdot 3 + 2^7 \cdot 3^2 + \dots + 3^9$$

$$= \frac{2^9 \left( \left( \frac{3}{2} \right)^{10} - 1 \right)}{\frac{3}{2} - 1} = 3^{10} - 2^{10}$$

$$\text{Now, } 3^{10} - 2^{10} = (3^5 - 2^5)(3^5 + 2^5)$$

$$= (211)(275)$$

$$= (35 \times 6 + 1)(45 \times 6 + 5)$$

$$= 6\lambda + 5$$

Remainder is 5.

9. Let  $f(x)$  be a polynomial function such that  $f(x) + f'(x) + f''(x) = x^5 + 64$ . Then, the value

$$\text{of } \lim_{x \rightarrow 1} \frac{f(x)}{x-1}$$

- (A) -15 (B) -60  
 (C) 60 (D) 15

Official Ans. by NTA (A)

Ans. (A)

Sol.  $\text{Lt}_{x \rightarrow 1} \frac{f(x)}{x-1} = f'(1)$  (and  $f(1) = 0$ )

$$f(x) + f'(x) + f''(x) = x^5 + 64$$

$$f'(x) + f''(x) + f'''(x) = 5x^4$$

$$f''(x) + f'''(x) + f^{iv}(x) = 20x^3$$

$$f'''(x) + f^{iv}(x) + f^v(x) = 60x^2$$

$$\therefore f^v(x) - f''(x) = 60x^2 - 20x^3$$

$$\Rightarrow 120 - f''(1) = 40 \Rightarrow f''(1) = 80$$

$$\text{Also } f(1) + f'(1) + f''(1) = 65 \Rightarrow f'(1) = -15. \text{ Ans.}$$

10. Let  $E_1$  and  $E_2$  be two events such that the conditional probabilities  $P(E_1|E_2) = \frac{1}{2}$ ,

$$P(E_2|E_1) = \frac{3}{4} \text{ and } P(E_1 \cap E_2) = \frac{1}{8}. \text{ Then:}$$

- (A)  $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$
- (B)  $P(E'_1 \cap E'_2) = P(E'_1) \cdot P(E_2)$
- (C)  $P(E_1 \cap E'_2) = P(E_1) \cdot P(E_2)$
- (D)  $P(E'_1 \cap E_2) = P(E_1) \cdot P(E_2)$

Official Ans. by NTA (C)

Ans. (C)

Sol.

$$(A) P(E_1) \cdot P(E_2) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24} \neq P(E_1 \cap E_2)$$

$$(B) P(E'_1 \cap E'_2) = 1 - P(E_1 \cup E_2) \\ = 1 - (P(E_1) + P(E_2) - P(E_1 \cap E_2)) \\ = 1 - \left( \frac{1}{6} + \frac{1}{4} - \frac{1}{8} \right) = \frac{17}{24}$$

$$P(E'_1)P(E_2) = \frac{5}{6} \times \frac{1}{4} = \frac{5}{24}$$

$$(C) P(E_1 \cap E'_2) = P(E_1) - P(E_1 \cap E_2) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

$$(D) P(E'_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

11. Let  $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ . If M and N are two matrices

$$\text{given by } M = \sum_{k=1}^{10} A^{2k} \text{ and } N = \sum_{k=1}^{10} A^{2k-1} \text{ then}$$

$MN^2$  is

- (A) a non-identity symmetric matrix
- (B) a skew-symmetric matrix
- (C) neither symmetric nor skew-symmetric matrix
- (D) an identity matrix

Official Ans. by NTA (A)

Ans. (A)

Sol.  $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = -4I$$

$$A^3 = -4A$$

$$A^4 = (-4I)(-4I) = (-4)^2I$$

$$A^5 = (-4)^2A, \quad A^6 = (-4)^3I$$

$$M = \sum_{k=1}^{10} A^{2k} = A^2 + A^4 + \dots + A^{20}$$

$$= [-4 + (-4)^2 + (-4)^3 + \dots + (-4)^{20}]I$$

$$= -4\lambda I$$

$\Rightarrow$  M is symmetric matrix

$$N = \sum_{k=1}^{10} A^{2k-1} = A + A^3 + \dots + A^{19}$$

$$= A[1 + (-4) + (-4)^2 + \dots + (-4)^9]$$

$$= \lambda A \Rightarrow \text{skew symmetric}$$

$\Rightarrow N^2$  is symmetric matrix

$\Rightarrow MN^2$  is non identity symmetric matrix

12. Let  $g : (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function such that

$$\int \left( \frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c,$$

for all  $x > 0$ , where c is an arbitrary constant. Then.

(A) g is decreasing in  $\left(0, \frac{\pi}{4}\right)$

(B)  $g'$  is increasing in  $\left(0, \frac{\pi}{4}\right)$

(C)  $g + g'$  is increasing in  $\left(0, \frac{\pi}{2}\right)$

(D)  $g - g'$  is increasing in  $\left(0, \frac{\pi}{2}\right)$

Official Ans. by NTA (D)

Ans. (D)

Sol.

$$\int \left( \frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c$$

On differentiating both sides w.r.t. x, we get

$$\left( \frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right)$$

$$= \frac{(e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)}{(e^x + 1)^2}$$

$$(e^x + 1)x(\cos x - \sin x) + g(x)(e^x + 1 - xe^x)$$

$$= (e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)$$

$$\Rightarrow g'(x) = \cos x - \sin x$$

$$\Rightarrow g(x) = \sin x + \cos x + C$$

g(x) is increasing in (0, π/4)

$$g''(x) = -\sin x - \cos x < 0$$

⇒ g'(x) is decreasing function

$$\text{let } h(x) = g(x) + g'(x) = 2 \cos x + C$$

$$\Rightarrow h'(x) = g'(x) + g''(x) = -2 \sin x < 0$$

⇒ h is decreasing

$$\text{let } \phi(x) = g(x) - g'(x) = 2 \sin x + C$$

$$\Rightarrow \phi'(x) = g'(x) - g''(x) = 2 \cos x > 0$$

⇒ φ is increasing

Hence option D is correct.

13. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be two functions defined by  $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$  and

$$g(x) = \frac{1 - 2e^{2x}}{e^x}. \text{ Then, for which of the}$$

following range of  $\alpha$ , the inequality

$$f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right) \text{ holds?}$$

(A) (2, 3)                      (B) (-2, -1)

(C) (1, 2)                      (D) (-1, 1)

Official Ans. by NTA (A)

Ans. (A)

Sol.  $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$

$$\Rightarrow f'(x) = \frac{2x}{x^2 + 1} + e^{-x} > 0 \quad \forall x \in \mathbb{R}$$

⇒ f is strictly increasing

$$g(x) = \frac{1 - 2e^{2x}}{e^x} = e^{-x} - 2e^x$$

$$\Rightarrow g'(x) = -(2e^x + e^{-x}) < 0 \quad \forall x \in \mathbb{R}$$

⇒ g is decreasing

$$\text{Now } f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right)$$

$$\Rightarrow g\left(\frac{(\alpha-1)^2}{3}\right) > g\left(\alpha - \frac{5}{3}\right)$$

$$\Rightarrow \frac{(\alpha-1)^2}{3} < \alpha - \frac{5}{3}$$

$$\Rightarrow \alpha^2 - 5\alpha + 6 < 0$$

$$\Rightarrow (\alpha - 2)(\alpha - 3) < 0$$

$$\Rightarrow \alpha \in (2, 3)$$

14. Let  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$   $a_i > 0$ ,  $i = 1, 2, 3$  be a vector which makes equal angles with the coordinates axes OX, OY and OZ. Also, let the projection of  $\vec{a}$  on the vector  $3\hat{i} + 4\hat{j}$  be 7. Let

$\vec{b}$  be a vector obtained by rotating  $\vec{a}$  with  $90^\circ$ .

If  $\vec{a}$ ,  $\vec{b}$  and x-axis are coplanar, then projection

of a vector  $\vec{b}$  on  $3\hat{i} + 4\hat{j}$  is equal to

(A)  $\sqrt{7}$                       (B)  $\sqrt{2}$

(C) 2                          (D) 7

Official Ans. by NTA (B)

Ans. (B)

Sol.  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a} = \lambda \left( \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) = \frac{\lambda}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

Now projection of  $\vec{a}$  on  $\vec{b} = 7$

$$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 7$$

$$\frac{\lambda}{\sqrt{3}} \frac{(\hat{i} + \hat{j} + \hat{k}) \cdot (3\hat{i} + 4\hat{j})}{5} = 7$$

$$\lambda = 5\sqrt{3}$$

$$\vec{a} = 5(\hat{i} + \hat{j} + \hat{k})$$

$$\text{now } \vec{b} = 5\alpha(\hat{i} + \hat{j} + \hat{k}) + \beta(\hat{i})$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow 25\alpha(3) + 5\beta = 0$$

$$\Rightarrow 15\alpha + \beta = 0 \Rightarrow \beta = -15\alpha$$

$$\vec{b} = 5\alpha(-2\hat{i} + \hat{j} + \hat{k})$$

$$|\vec{b}| = 5\sqrt{3}$$

$$\Rightarrow \alpha = \pm \frac{1}{\sqrt{2}}$$

$$\vec{b} = \pm \frac{5}{\sqrt{2}}(-2\hat{i} + \hat{j} + \hat{k})$$

Projection of  $\vec{b}$  on  $3\hat{i} + 4\hat{j}$  is

$$\frac{\vec{b} \cdot (3\hat{i} + 4\hat{j})}{5} = \pm \frac{5}{\sqrt{2}} \left( \frac{-6 + 4}{5} \right) = \pm \sqrt{2}$$

15. Let  $y = y(x)$  be the solution of the differential equation  $(x + 1)y' - y = e^{3x}(x + 1)^2$ , with

$$y(0) = \frac{1}{3}. \text{ Then, the point } x = -\frac{4}{3} \text{ for the curve}$$

$y = y(x)$  is:

- (A) not a critical point
- (B) a point of local minima
- (C) a point of local maxima
- (D) a point of inflection

Official Ans. by NTA (B)

Ans. (B)

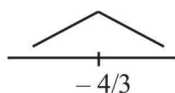
Sol.  $(x + 1)dy - y dx = e^{3x}(x + 1)^2$

$$\frac{(x + 1)dy - y dx}{(x + 1)^2} = e^{3x}$$

$$d\left(\frac{y}{x + 1}\right) = e^{3x} \Rightarrow \frac{y}{x + 1} = \frac{e^{3x}}{3} + C$$

$$\left(0, \frac{1}{3}\right) \Rightarrow C = 0 \Rightarrow y = \frac{(x + 1)e^{3x}}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}((x + 1)3e^{3x} + e^{3x}) = \frac{3^{3x}}{3}(3x + 4)$$



Clearly,  $x = -\frac{4}{3}$  is point of local minima

16. If  $y = m_1x + c_1$  and  $y = m_2x + c_2$ ,  $m_1 \neq m_2$  are two common tangents of circle  $x^2 + y^2 = 2$  and parabola  $y^2 = x$ , then the value of  $8|m_1m_2|$  is equal to

- (A)  $3 + 4\sqrt{2}$
- (B)  $-5 + 6\sqrt{2}$
- (C)  $-4 + 3\sqrt{2}$
- (D)  $7 + 6\sqrt{2}$

Official Ans. by NTA (C)

Ans. (C)

Sol.  $C_1: x^2 + y^2 = 2$

$$C_2: y^2 = x$$

Let tangent to parabola be  $y = mx + \frac{1}{4m}$ .

It is also a tangent of circle so distance from centre of circle  $(0, 0)$  will be  $\sqrt{2}$ .

$$\left| \frac{\frac{1}{4m}}{\sqrt{1 + m^2}} \right| = \sqrt{2} \Rightarrow 1 = 32m^2 + 32m^4$$

by solving

$$m^2 = \frac{3\sqrt{2} - 4}{8}, \quad m^2 = \frac{-3\sqrt{2} - 4}{8} \text{ (rejected)}$$

$$m = \pm \sqrt{\frac{3\sqrt{2} - 4}{8}}$$

$$\text{so, } 8|m_1m_2| = 3\sqrt{2} - 4$$

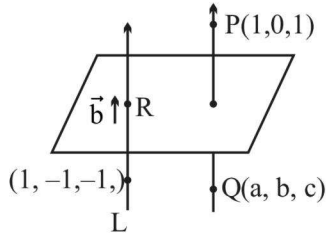
17. Let Q be the mirror image of the point  $P(1, 0, 1)$  with respect to the plane  $S: x + y + z = 5$ . If a line L passing through  $(1, -1, -1)$ , parallel to the line PQ meets the plane S at R, then  $QR^2$  is equal to:

- (A) 2
- (B) 5
- (C) 7
- (D) 11

Official Ans. by NTA (B)

Ans. (B)

Sol.



Let parallel vector of  $L = \vec{b}$

mirror image of  $Q$  on given plane  $x+y+z=5$

$$\frac{a-1}{1} = \frac{b-0}{1} = \frac{c-1}{1} = \frac{-2(2-5)}{3}$$

$a = 3, b = 2, c = 3$

$Q = (3, 2, 3)$

$$\therefore \vec{b} \parallel \overrightarrow{PQ}$$

so,  $\vec{b} = (1, 1, 1)$

Equation of line

$$L : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z+1}{1}$$

Let point  $R, (\lambda+1, \lambda-1, \lambda-1)$

lying on plane  $x + y + z = 5,$

so,  $3\lambda - 1 = 5$

$\Rightarrow \lambda = 2$

Point  $R$  is  $(3, 1, 1)$

$QR^2 = 5$  Ans.

18. If the solution curve  $y = y(x)$  of the differential equation  $y^2 dx + (x^2 - xy + y^2) dy = 0$ , which passes through the point  $(1, 1)$  and intersects the line  $y = \sqrt{3}x$  at the point  $(\alpha, \sqrt{3}\alpha)$ , then value of  $\log_e(\sqrt{3}\alpha)$  is equal to

- (A)  $\frac{\pi}{3}$  (B)  $\frac{\pi}{2}$   
 (C)  $\frac{\pi}{12}$  (D)  $\frac{\pi}{6}$

Official Ans. by NTA (C)

Ans. (C)

Sol.  $y^2 dx - xy dy = -(x^2 + y^2) dy$   
 $y(y dx - x dy) = -(x^2 + y^2) dy$   
 $-y(x dx - y dy) = -(x^2 + y^2) dy$

$$\frac{xdy - ydx}{x^2} = \left(1 + \frac{y^2}{x^2}\right) \frac{dy}{y}$$

$$\Rightarrow \frac{d(y/x)}{1 + \frac{y^2}{x^2}} = \frac{dy}{y}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln y + C$$

$$(\alpha, \sqrt{3}\alpha) \Rightarrow \frac{\pi}{3} = \ln(\sqrt{3}\alpha) + \frac{\pi}{4}$$

$$\therefore \ln(\sqrt{3}\alpha) = \frac{\pi}{12}$$

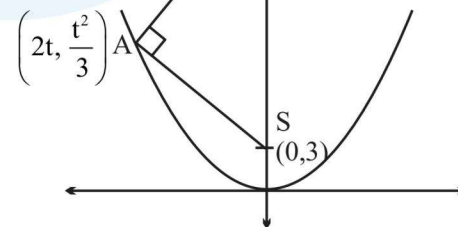
19. Let  $x = 2t, y = \frac{t^2}{3}$  be a conic. Let  $S$  be the focus and  $B$  be the point on the axis of the conic such that  $SA \perp BA$ , where  $A$  is any point on the conic. If  $k$  is the ordinate of the centroid of  $\Delta SAB$ , then  $\lim_{t \rightarrow 1} k$  is equal to

- (A)  $\frac{17}{18}$  (B)  $\frac{19}{18}$   
 (C)  $\frac{11}{18}$  (D)  $\frac{13}{18}$

Official Ans. by NTA (D)

Ans. (D)

Sol.



parabola  $x^2 = 12y$   
 $SA \perp SB$

so,  $m_{AS} \cdot m_{AB} = -1$

$$\frac{\left(3 - \frac{t^2}{3}\right)}{(0-2t)} \cdot \frac{\left(\alpha - \frac{t^2}{3}\right)}{(0-2t)} = -1$$

by solving

$$3\alpha = \frac{27t^2 + t^4}{t^2 - 9}$$

ordinate of centroid of  $\Delta SAB = K = \frac{\alpha + \frac{t^2}{3} + 3}{3}$

$$k = \frac{9 + 3\alpha + t^2}{9}$$

$$\lim_{t \rightarrow 1} k = \lim_{t \rightarrow 1} \frac{1}{9} \left( 9 + t^2 + \frac{27t^2 + t^4}{t^2 - 9} \right) = \frac{13}{18}$$

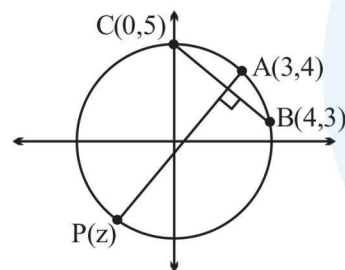
20. Let a circle C in complex plane pass through the points  $z_1 = 3 + 4i$ ,  $z_2 = 4 + 3i$  and  $z_3 = 5i$ . If  $z (\neq z_1)$  is a point on C such that the line through z and  $z_1$  is perpendicular to the line through  $z_2$  and  $z_3$ , then  $\arg(z)$  is equal to :

(A)  $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right) - \pi$       (B)  $\tan^{-1}\left(\frac{24}{7}\right) - \pi$

(C)  $\tan^{-1}(3) - \pi$       (D)  $\tan^{-1}\left(\frac{3}{4}\right) - \pi$

Official Ans. by NTA (B)

Ans. (B)



Sol.

Slope of BC =  $\frac{3-5}{4-0} = -\frac{1}{2}$

Slope of AP = 2

equation of AP :  $y - 4 = 2(x - 3)$

$\Rightarrow y = 2(x - 1)$

P lies on circle  $x^2 + y^2 = 25$

$\Rightarrow x^2 + (2(x - 1))^2 = 25$

$\Rightarrow x = -\frac{7}{5}$  and  $y = -\frac{24}{5}$

$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{24}{7}\right) - \pi$

SECTION-B

1. Let  $C_r$  denote the binomial coefficient of  $x^r$  in the expansion of  $(1 + x)^{10}$ . If  $\alpha, \beta \in \mathbb{R}$ .  $C_1 + 3 \cdot 2C_2 + 5 \cdot 3C_3 + \dots$  upto 10 terms

$$= \frac{\alpha \times 2^{11}}{2^\beta - 1} \left( C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots \text{upto 10 terms} \right)$$

then the value of  $\alpha + \beta$  is equal to

Official Ans. by NTA (286)

Ans. (BONUS)

Sol.  $(1 + x)^{10} = C_0 + C_1x + C_2x^2 + \dots + C_{10}x^{10}$

Differentiating

$$10(1 + x)^9 = C_1 + 2C_2x + 3C_3x^2 + \dots + 10C_{10}x^9$$

replace  $x \rightarrow x^2$

$$10(1 + x^2)^9 = C_1 + 2C_2x^2 + 3C_3x^4 + \dots + 10C_{10}x^{18}$$

$$10 \cdot x(1 + x^2)^9 = C_1x + 2C_2x^3 + 3C_3x^5 + \dots + 10C_{10}x^{19}$$

Differentiating

$$10 \left( (1 + x^2)^9 \cdot 1 + x \cdot 9(1 + x^2)^8 \cdot 2x \right)$$

$$= C_1x + 2 \cdot 2C_2 \cdot 3x^3 + 3 \cdot 5 \cdot C_3x^4 + \dots + 10 \cdot 19C_{10}x^{18}$$

putting  $x = 1$

$$10(2^9 + 18 \cdot 2^8)$$

$$= C_1 + 3 \cdot 2 \cdot C_2 + 5 \cdot 3 \cdot C_3 + \dots + 19 \cdot 10 \cdot C_{10}$$

$$C_1 + 3 \cdot 2 \cdot C_2 + \dots + 19 \cdot 10 \cdot C_{10}$$

$$= 10 \cdot 2^9 \cdot 10 = 100 \cdot 2^9$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_9}{11} + \frac{C_{10}}{11} = \frac{2^{11} - 1}{11}$$

10<sup>th</sup> term 11<sup>th</sup> term

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_9}{11} = \frac{2^{11} - 2}{11}$$

Now,  $100 \cdot 2^9 = \frac{\alpha \cdot 2^{11}}{2^\beta - 1} \left( \frac{2^{11} - 2}{11} \right)$

Eqn. of form  $y = k(2^x - 1)$ .

It has infinite solutions even if we take  $x, y \in \mathbb{N}$ .



2. The number of 3-digit odd numbers, whose sum of digits is a multiple of 7, is \_\_\_\_\_.

Official Ans. by NTA (63)

Ans. (63)

Sol.  $x y z \leftarrow$  odd number

$$z = 1, 3, 5, 7, 9$$

$$x+y+z = 7, 14, 21 \text{ [sum of digit multiple of 7]}$$

$$\begin{matrix} x & + & y \\ \text{1 to 9} & & \text{0 to 9} \end{matrix} = 6, 4, 2, 13, 11, 9, 7, 5, 20, 18, 16, 14, 12$$

$$x + y = 6 \Rightarrow (1,5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)$$

$$\rightarrow \text{T.N.} = 6$$

$$x + y = 4 \Rightarrow (1,3), (2, 2), (3, 1), (4,0)$$

$$\rightarrow \text{T.N.} = 4$$

$$x + y = 2 \Rightarrow (1,1), (2,0)$$

$$\rightarrow \text{T.N.} = 2$$

$$x + y = 13 \Rightarrow (4,9), (5,8), (6,7), (7,6), (8,5), (9,4)$$

$$\rightarrow \text{T.N.} = 6$$

$$x + y = 11 \Rightarrow (2,9), (3,8), (4,7), (5,6), (6,5), (6,5), (7,4), (8,3), (9,2)$$

$$\rightarrow \text{T.N.} = 8$$

$$x + y = 9 \Rightarrow (1,8), (2,7), (3,6), (4,5), (5,4), \dots, (8,1), (9,0)$$

$$\rightarrow \text{T.N.} = 9$$

$$x + y = 7 \Rightarrow (1,6), (2,5), (3,4), \dots, (8, 1), (7,0)$$

$$\rightarrow \text{T.N.} = 7$$

$$x + y = 5 \Rightarrow (1,4), (2,3), (3, 2), (4,1), (5,0)$$

$$\rightarrow \text{T.N.} = 5$$

$$x + y = 20 \Rightarrow \text{Not possible}$$

$$x + y = 18 \Rightarrow (9,9) \rightarrow \text{T.N.} = 1$$

$$x + y = 16 \Rightarrow (7,9), (8,8), (9,7)$$

$$\rightarrow \text{T.N.} = 3$$

$$x + y = 14 \Rightarrow (5,9), (6,8), (7,7), (8,6), (9,5)$$

$$\rightarrow \text{T.N.} = 5$$

$$x + y = 12 \Rightarrow (3,9), (4,8), (5,7), (6,6) \dots (9,3)$$

$$\rightarrow \text{T.N.} = 7$$

3. Let  $\theta$  be the angle between the vectors  $\vec{a}$  and  $\vec{b}$ ,

$$\text{where } |\vec{a}| = 4, |\vec{b}| = 3 \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right). \text{ Then}$$

$$\left|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})\right|^2 + 4(\vec{a} \cdot \vec{b})^2 \text{ is equal to } \underline{\hspace{2cm}}$$

Official Ans. by NTA (576)

Ans. (576)

$$\text{Sol. } |\vec{a}| = 4, |\vec{b}| = 3 \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$$

$$\left|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})\right|^2 + 4(\vec{a} \cdot \vec{b})^2$$

$$\left|\vec{a} \times \vec{b} - \vec{b} \times \vec{a}\right|^2 + 4a^2b^2 \cos^2 \theta$$

$$2\left|\vec{a} \times \vec{b}\right|^2 + 4a^2b^2 \cos^2 \theta$$

$$4a^2b^2 \sin^2 \theta + 4a^2b^2 \cos^2 \theta$$

$$4a^2b^2 = 4 \times 16 \times 9 = 576$$

4. Let the abscissae of the two points P and Q be the roots of  $2x^2 - rx + p = 0$  and the ordinates of P and Q be the roots of  $x^2 - sx - q = 0$ . If the equation of the circle described on PQ as diameter is  $2(x^2 + y^2) - 11x - 14y - 22 = 0$ , then  $2r + s - 2q + p$  is equal to

Official Ans. by NTA (7)

Ans. (7)

$$\text{Sol. } 2x^2 - rx + p = 0 \begin{cases} x_1 \\ x_2 \end{cases}$$

$$y^2 - sy - q = 0 \begin{cases} y_1 \\ y_2 \end{cases}$$

Equation of the circle with PQ as diameter is

$$2(x^2 + y^2) - rx - 2sy + p - 2q = 0$$

on comparing with the given equation

$$r = 11, s = 7$$

$$p - 2q = -22$$

$$\therefore 2r + s - 2q + p = 22 + 7 - 22 = 7$$

5. The number of values of  $x$  in the interval

$$\left(\frac{\pi}{4}, \frac{7\pi}{4}\right) \text{ for which } 14\operatorname{cosec}^2x - 2\sin^2x = 21$$

$$-4\cos^2x \text{ holds, is } \underline{\hspace{2cm}}$$

Official Ans. by NTA (4)

Ans. (4)

$$\text{Sol. } x \in \left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$$

$$14\operatorname{cosec}^2x - 2\sin^2x = 21 - 4\cos^2x$$

$$= 21 - 4(1 - \sin^2x)$$

$$= 17 + 4\sin^2x$$

$$14\operatorname{cosec}^2x - 6\sin^2x = 17$$

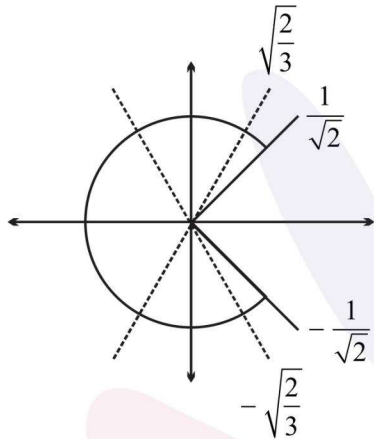
$$\text{let } \sin^2x = p$$

$$\frac{14}{p} - 6p = 17 \Rightarrow 14 - 6p^2 = 17p$$

$$6p^2 + 17p - 14 = 0$$

$$p = -3.5, \frac{2}{3} \Rightarrow \sin^2 x = \frac{2}{3}$$

$$\Rightarrow \sin x = \pm \sqrt{\frac{2}{3}}$$



∴ Total 4 solutions

6. For a natural number  $n$ , let  $a_n = 19^n - 12^n$ . Then,

the value of  $\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8}$  is

**Official Ans. by NTA (4)**

**Ans. (4)**

**Sol.**  $a_n = 19^n - 12^n$

$$\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} = \frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57\alpha_8}$$

$$= \frac{19^9(31-19) - 12^9(31-12)}{57\alpha_8}$$

$$= \frac{19^9 \cdot 12 - 12^{10} \cdot 19}{57\alpha_8}$$

$$= \frac{12 \cdot 19(19^8 - 12^8)}{57\alpha_8} = 4$$

7. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by

$$f(x) = \left( 2 \left( 1 - \frac{x^{25}}{2} \right) (2 + x^{25}) \right)^{\frac{1}{50}}$$

. If the function  $g(x) = f(f(f(x))) + f(f(x))$ , the the greatest integer less than or equal to  $g(1)$  is \_\_\_\_\_

**Official Ans. by NTA (2)**

**Ans. (2)**

**Sol.**  $f(x) = \left[ 2 \left( 1 - \frac{x^{25}}{2} \right) (2 + x^{25}) \right]^{\frac{1}{50}}$

$$f(x) = \left[ (2 - x^{25})(2 + x^{25}) \right]^{\frac{1}{50}} = (4 - x^{50})^{1/50}$$

$$f(f(x)) = \left( 4 - \left( (4 - x^{50})^{1/50} \right)^{50} \right)^{1/50} = x$$

$$g(x) = f(f(f(x))) + f(f(x))$$

$$= f(x) + x$$

$$g(1) = f(1) + 1 = 3^{1/50} + 1$$

$$[g(1)] = [3^{1/50} + 1] = 2$$

8. Let the lines

$$L_1 : \vec{r} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k}), \lambda \in \mathbb{R}$$

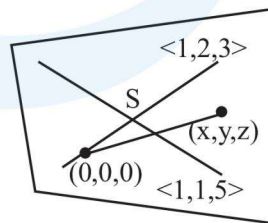
$$L_2 : \vec{r} = (\hat{i} + 3\hat{j} + \hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k}); \mu \in \mathbb{R}$$

intersect at the point  $S$ . If a plane  $ax + by - z + d = 0$  passes through  $S$  and is parallel to both the lines  $L_1$  and  $L_2$ , then the value of  $a + b + d$  is equal to \_\_\_\_\_

**Official Ans. by NTA (5)**

**Ans. (5)**

**Sol.** Both the lines lie in the same plane



∴ equation of the plane

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 7x - 2y - z = 0$$

$$\therefore a + b + d = 5$$

9. Let A be a  $3 \times 3$  matrix having entries from the set  $\{-1, 0, 1\}$ . The number of all such matrices A having sum of all the entries equal to 5, is \_\_\_\_\_

**Official Ans. by NTA (414)**

**Ans. (414)**

**Sol. Case-I:**  $1 \rightarrow 7$  times  
and  $-1 \rightarrow 2$  times

$$\text{number of possible matrix} = \frac{9!}{7!2!} = 36$$

**Case-II:**  $1 \rightarrow 6$  times,  
 $-1 \rightarrow 1$  times  
and  $0 \rightarrow 2$  times

$$\text{number of possible matrix} = \frac{9!}{6!2!} = 252$$

**Case-III:**  $1 \rightarrow 5$  times,  
and  $0 \rightarrow 4$  times

$$\text{number of possible matrix} = \frac{9!}{5!4!} = 126$$

Hence total number of all such matrix A  
= 414

10. The greatest integer less than or equal to the sum of first 100 terms of the sequence

$$\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \dots$$

is equal to

**Official Ans. by NTA (98)**

**Ans. (98)**

**Sol.**  $\frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \dots$

$$\left(1 - \frac{2}{3}\right) + \left(1 - \frac{4}{9}\right) + \left(1 - \frac{8}{27}\right) + \left(1 - \frac{16}{81}\right) \dots 100 \text{ terms}$$

$$100 - \left[ \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots \right]$$

$$100 - \frac{\frac{2}{3} \left(1 - \left(\frac{2}{3}\right)^{100}\right)}{1 - \frac{2}{3}}$$

$$100 - 2 \left(1 - \left(\frac{2}{3}\right)^{100}\right)$$

$$S = 98 + 2 \left(\frac{2}{3}\right)^{100}$$

$$\Rightarrow [S] = 98$$