FINAL JEE–MAIN EXAMINATION – JANUARY, 2023

(Held On Sunday 29th January, 2023) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

61. The statement $B \Rightarrow ((\sim A) \vee B)$ is equivalent to :

(1) $B \Rightarrow (A \Rightarrow B)$

(2) $A \Rightarrow (A \Leftrightarrow B)$

(3) $A \Rightarrow ((\sim A) \Rightarrow B)$

(4) $B \Rightarrow ((\sim A) \Rightarrow B)$

Official Ans. by NTA (1,3,4)

Ans. (1 or 3 or 4)

Sol.

A	B	~A	~A ∨ B	$B \Rightarrow ((\sim A) \vee B)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

$A \Rightarrow B$	$\sim A \Rightarrow B$	$B \Rightarrow (A \Rightarrow B)$	$A \Rightarrow ((\sim A) \Rightarrow B)$	$B \Rightarrow ((\sim A) \Rightarrow B)$
T	T	T	T	T
F	T	T	T	T
T	T	T	T	T
T	F	T	T	T

62. Shortest distance between the lines

$\frac{x-1}{2}=\frac{y+8}{-7}=\frac{z-4}{5}$ and $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-6}{-3}$ is

(1) $2\sqrt{3}$

(2) $4\sqrt{3}$

(3) $3\sqrt{3}$

(4) $5\sqrt{3}$

Official Ans. by NTA (2)

Ans. (2)

Sol. $\frac{x-1}{2}=\frac{y+8}{-7}=\frac{z-4}{5}$ $\quad \vec{a}=\hat{i}-8\hat{j}+4\hat{k}$

$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-6}{-3}$ $\quad \vec{b}=\hat{i}+2\hat{j}+6\hat{k}$

$\vec{p}=2\hat{i}-7\hat{j}+5\hat{k},\ \vec{q}=2\hat{i}+\hat{j}-3\hat{k}$

$$\vec{p}\times\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -7 & 5\\ 2 & 1 & -3\end{vmatrix}$$

$=\hat{i}(16)-\hat{j}(-16)+\hat{k}(16)$

$=16(\hat{i}+\hat{j}+\hat{k})$

$$d=\left|\frac{(a-b)\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|=\left|\frac{(-10\hat{j}-2\hat{k})\cdot 16(\hat{i}+\hat{j}+\hat{k})}{16\sqrt{3}}\right|$$

$=\left|\frac{-12}{\sqrt{3}}\right|=4\sqrt{3}$

63. If $\vec{a}=\hat{i}+2\hat{k},\ \vec{b}=\hat{i}+\hat{j}+\hat{k},\ \vec{c}=7\hat{i}-3\hat{k}+4\hat{k},$ $\vec{r}\times\vec{b}+\vec{b}\times\vec{c}=\vec{0}$ and $\vec{r}.\vec{a}=0$ then $\vec{r}.\vec{c}$ is equal to :

(1) 34 (2) 12

(3) 36 (4) 30

Official Ans. by NTA (1)

Ans. (1)

Sol. $\vec{r}\times\vec{b}-\vec{c}\times\vec{b}=0$

$\Rightarrow (\vec{r}-\vec{c})\times\vec{b}=0$

$\Rightarrow \vec{r}-\vec{c}=\lambda\vec{b}$

$\Rightarrow \vec{r}=\vec{c}+\lambda\vec{b}$

And given that $\vec{r}\cdot\vec{a}=0$

$\Rightarrow (\vec{c}+\lambda\vec{b})\cdot\vec{a}=0$

$\Rightarrow \vec{c}\cdot\vec{a}+\lambda\vec{b}\cdot\vec{a}=0$

$\Rightarrow \lambda=\frac{-\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}$

Now $\vec{r}\cdot\vec{c}=(\vec{c}+\lambda\vec{b})\cdot\vec{c}$

$=\left(\vec{c}-\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\vec{b}\right)\cdot\vec{c}$

$=|\vec{c}|-\left(\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\right)(\vec{b}\cdot\vec{c})$

$=74-\left[\frac{15}{3}\right]8$

$=74-40=34$

64. Let $S=\{w_1,w_2,....\}$ be the sample space associated to a random experiment. Let $P(w_n)=\frac{P(w_{n-1})}{2}, n\geq 2$. Let $A=\{2k+3\ell;\ k,\ell\in\mathbb{N}\}$ and $B=\{w_n;n\in A\}$. Then P(B) is equal to

(1) $\frac{3}{32}$ (2) $\frac{3}{64}$

(3) $\frac{1}{16}$ (4) $\frac{1}{32}$

Official Ans. by NTA (2)

Ans. (2)

Sol. Let $P(w_1)=\lambda$ then $P(w_2)=\frac{\lambda}{2}$ … $P(w_n)=\frac{\lambda}{2^{n-1}}$

As $\sum_{k=1}^{\infty}P(w_k)=1 \Rightarrow \frac{\lambda}{1-\frac{1}{2}}=1 \Rightarrow \lambda=\frac{1}{2}$

So, $P(w_n)=\frac{1}{2^n}$

$A=\{2k+3\ell;k,\ell\in\mathbb{N}\}=\{5, 7, 8, 9, 10 .....\}$

$B=\{w_n : n\in A\}$

$B=\{w_5, w_7, w_8, w_9, w_{10}, w_{11}, ...\}$

$A=\mathbb{N}-\{1,2,3,4,6\}$

$\therefore P(B)=1-[P(w_1)+P(w_2)+P(w_3)+P(w_4)+P(w_6)]$

$=1-\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{64}\right]$

$=1-\frac{32+16+8+4+1}{64}=\frac{3}{64}$

65. The value of the integral $\int_1^2\left(\frac{t^4+1}{t^6+1}\right)dt$ is :

(1) $\tan^{-1}\frac{1}{2}+\frac{1}{3}\tan^{-1}8-\frac{\pi}{3}$

(2) $\tan^{-1}2-\frac{1}{3}\tan^{-1}8+\frac{\pi}{3}$

(3) $\tan^{-1}2+\frac{1}{3}\tan^{-1}8-\frac{\pi}{3}$

(4) $\tan^{-1}\frac{1}{2}-\frac{1}{3}\tan^{-1}8+\frac{\pi}{3}$

Official Ans. by NTA (3)

Ans. (3)

Sol. $I=\int_1^2\left(\frac{t^4+1}{t^6+1}\right)dt$

$=\int_1^2\frac{(t^4+1-t^2)+t^2}{(t^2+1)(t^4-t^2+1)}dt$

$=\int_1^2\left(\frac{1}{t^2+1}+\frac{t^2}{t^6+1}\right)dt$

$=\int_1^2\left(\frac{1}{t^2+1}+\frac{1}{3}\frac{3t^2}{(t^3)^2+1}\right)dt$

$=\tan^{-1}(t)+\frac{1}{3}\tan^{-1}(t^3)\Big|_1^2$

$=(\tan^{-1}(2)-\tan^{-1}(1))+\frac{1}{3}(\tan^{-1}(2^3)-\tan^{-1}(1^3))$

$=\tan^{-1}(2)+\frac{1}{3}\tan^{-1}(8)-\frac{\pi}{3}$

66. Let K be the sum of the coefficients of the odd powers of x in the expansion of $(1+x)^{99}$. Let a be the middle term in the expansion of $\left(2+\frac{1}{\sqrt{2}}\right)^{200}$. If $\frac{{}^{200}C_{99}K}{a}=\frac{2^{\ell}m}{n}$, where m and n are odd numbers, then the ordered pair (ℓ, n) is equal to :

(1) (50, 51)

(2) (51, 99)

(3) (50, 101)

(4) (51, 101)

Official Ans. by NTA (3)

Ans. (3)

Sol. In the expansion of

$$(1+x)^{99} = C_0 + C_1x + C_2x^2 + ... + C_{99}x^{99}$$

$$K = C_1 + C_3 + ... + C_{99} = 2^{98}$$

$a \Rightarrow$ Middle in the expansion of $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$

$$T_{\frac{200}{2}+1} = {}^{200}C_{100}\,(2)^{100}\left(\frac{1}{\sqrt{2}}\right)^{100}$$

$$= {}^{200}C_{100}\,.2^{50}$$

So, $\frac{{}^{200}C_{99} \times 2^{98}}{{}^{200}C_{100} \times 2^{50}} = \frac{100}{101} \times 2^{48}$

So, $\frac{25}{101} \times 2^{50} = \frac{m}{n} 2^{\ell}$

$\therefore$ m, n are odd so

(ℓ, n) become (50, 101) Ans.

67. Let f and g be twice differentiable functions on R such that

$f''(x) = g''(x) + 6x$

$f'(1) = 4g'(1) - 3 = 9$

$f(2) = 3g(2) = 12$

Then which of the following is NOT true ?

(1) $g(-2) - f(-2) = 20$

(2) If $-1 < x < 2$, then $|f(x) - g(x)| < 8$

(3) $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1|$

(4) There exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

Official Ans. by NTA (2)

Ans. (2)

Sol. $f''(x) = g''(x) + 6x$ ...(1)

$f'(1) = 4g'(1) - 3 = 9$ ...(2)

$f(2) = 3g(2) = 12$ ...(3)

By integrating (1)

$$f'(x) = g'(x) + 6\frac{x^2}{2} + C$$

At $x = 1$,

$$f'(1) = g'(1) + 3 + C$$

$\Rightarrow 9 = 4 + 3 + C \Rightarrow C = 3$

$\therefore f'(x) = g'(x) + 3x^2 + 3$

Again by integrating,

$$f(x) = g(x) + \frac{3x^3}{3} + 3x + D$$

At $x = 2$,

$f(2) = g(2) + 8 + 3(2) + D$

$\Rightarrow 12 = 4 + 8 + 6 + D \Rightarrow D = -6$

So, $f(x) = g(x) + x^3 + 3x - 6$

$\Rightarrow f(x) - g(x) = x^3 + 3x - 6$

At $x = -2$,

$\Rightarrow g(-2) - f(-2) = 20$ (Option (1) is true)

Now, for $-1 < x , 2$

$h(x) = f(x) - g(x) = x^3 + 3x - 6$

$\Rightarrow h'(x) = 3x^2 + 3$

$\Rightarrow h(x)\uparrow$

So, $h(-1) < h(x) < h(2)$

$\Rightarrow -10 < h(x) < 8$

$\Rightarrow |h(x)| < 10$ (option (2) is NOT true)

Now, $h'(x) = f'(x) - g'(x) = 3x^2 + 3$

If $|h'(x)| < 6 \Rightarrow |3x^2 + 3| < 6$

$\Rightarrow 3x^2 + 3 < 6$

$\Rightarrow x^2 < 1$

$\Rightarrow -1 < x < 1$ (option (3) is True)

If $x \in (-1, 1)$ $|f'(x) - g'(x)| < 6$

option (3) is true and now to solve

$f(x) - g(x) = 0$

$\Rightarrow x^3 + 3x - 6 = 0$

$h(x) = x^3 + 3x - 6$

here, $h(1) = -ve$ and $h\left(\frac{3}{2}\right) = +ve$

So there exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

(option (4) is true)

68. The set of all values of $t \in \mathbb{R}$, for which the matrix

$$\begin{bmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$ is

invertible, is

(1) $\left\{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\right\}$ (2) $\left\{k\pi + \frac{\pi}{4}, k \in \mathbb{Z}\right\}$

(3) $\{k\pi, k \in \mathbb{Z}\}$ (4) $\mathbb{R}$

Official Ans. by NTA (4)

Ans. (4)

Sol. If its invertible, then determinant value $\neq 0$

So ,

$$\begin{vmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{vmatrix} \neq 0$$

$$\Rightarrow e^t \cdot e^{-t} \cdot e^{-t} \begin{vmatrix} 1 & \sin t - 2\cos t & -2\sin t - \cos t \\ 1 & 2\sin t + \cos t & \sin t - 2\cos t \\ 1 & \cos t & \sin t \end{vmatrix} \neq 0$$

Applying, $R_1 \to R_1 - R_2$ then $R_2 \to R_2 - R_3$

We get

$$e^{-t} \begin{vmatrix} 0 & -\sin t - \cos t & -3\sin t + \cos t \\ 0 & 2\sin t & -2\cos t \\ 1 & \cos t & \sin t \end{vmatrix} \neq 0$$

By expanding we have,

$$e^{-t} \times 1\left(2\sin t\cos t + 6\cos^2 t + 6\sin^2 t - 2\sin t\cos t\right) \neq 0$$

$$\Rightarrow e^{-t} \times 6 \neq 0$$

for $\forall\, t \in \mathbb{R}$

69. The area of the region

$$A = \left\{(x,y) : |\cos x - \sin x| \le y \le \sin x, 0 \le x \le \frac{\pi}{2}\right\}$$

(1) $1 - \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{5}}$ (2) $\sqrt{5} + 2\sqrt{2} - 4.5$

(3) $\frac{3}{\sqrt{5}} - \frac{3}{\sqrt{2}} + 1$ (4) $\sqrt{5} - 2\sqrt{2} + 1$

Official Ans. by NTA (4)

Ans. (4)

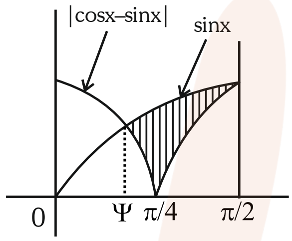
Sol. $|\cos x - \sin x| \le y \le \sin x$

Intersection point of $\cos x - \sin x = \sin x$

$$\Rightarrow \tan x = \frac{1}{2}$$

Let $\psi = \tan^{-1}\frac{1}{2}$

So, $\tan\psi = \frac{1}{2}$, $\sin\psi = \frac{1}{\sqrt{5}}$, $\cos\psi = \frac{2}{\sqrt{5}}$



$$\text{Area} = \int_{\psi}^{\pi/2} (\sin x - |\cos x - \sin x|)dx$$

$$= \int_{\psi}^{\pi/4} (\sin x - (\cos x - \sin x))dx$$

$$+ \int_{\pi/4}^{\pi/2} (\sin x - (\sin x - \cos x))dx$$

$$= \int_{\psi}^{\pi/4} (2\sin x - \cos x)dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$$

$$= [-2\cos x - \sin x]_{\psi}^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$$

$$= -\sqrt{2} - \frac{1}{\sqrt{2}} + 2\cos\psi + \sin\psi + \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= -\sqrt{2} - \frac{1}{\sqrt{2}} + 2\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}}\right) + 1 - \frac{1}{\sqrt{2}}$$

$$= \sqrt{5} - 2\sqrt{2} + 1$$

70. The set of all values of λ for which the equation $\cos^2 2x - 2\sin^4 x - 2\cos^2 x = \lambda$

(1) $[-2, -1]$ (2) $\left[-2, -\frac{3}{2}\right]$

(3) $\left[-1, -\frac{1}{2}\right]$ (4) $\left[-\frac{3}{2}, -1\right]$

Official Ans. by NTA (4)

Ans. (4)

Sol. $\lambda = \cos^2 2x - 2\sin^4 x - 2\cos^2 x$

convert all in to cos x.

$\lambda = (2\cos^2 x - 1)^2 - 2(1 - \cos^2 x)^2 - 2\cos^2 x$

$= 4\cos^4 x - 4\cos^2 x + 1 - 2(1 - 2\cos^2 x + \cos^4 x) - 2\cos^2 x$

$= 2\cos^4 x - 2\cos^2 x + 1 - 2$

$= 2\cos^4 x - 2\cos^2 x - 1$

$$= 2\left[\cos^4 x - \cos^2 x - \frac{1}{2}\right]$$

$$= 2\left[\left(\cos^2 x - \frac{1}{2}\right)^2 - \frac{3}{4}\right]$$

$$\lambda_{max} = 2\left[\frac{1}{4} - \frac{3}{4}\right] = 2 \times \left(-\frac{2}{4}\right) = -1 \text{ (max Value)}$$

$$\lambda_{min} = 2\left[0 - \frac{3}{4}\right] = -\frac{3}{2} \text{ (Minimum Value)}$$

So, Range $= \left[-\frac{3}{2}, -1\right]$

71. The letters of the word OUGHT are written in all possible ways and these words are arranged as in a dictionary, in a series. Then the serial number of the word TOUGH is :

(1) 89 (2) 84

(3) 86 (4) 79

Official Ans. by NTA (1)

Ans. (1)

Sol. Lets arrange the letters of OUGHT in alphabetical order.

G, H, O, T, U

Words starting with

$G - - - - - \rightarrow 4!$

$H - - - - - \rightarrow 4!$

$O - - - - - \rightarrow 4!$

$T\,G - - - \rightarrow 3!$

$T\,H - - - \rightarrow 3!$

$T\,O\,G - - \rightarrow 2!$

$T\,O\,H - - \rightarrow 2!$

$T\,O\,U\,G\,H \rightarrow 1!$

Total = 89

72. The plane $2x - y + z = 4$ intersects the line segment joining the points A(a, –2, 4) and B(2, b, –3) at the point C in the ratio 2 : 1 and the distance of the point C from the origin is $\sqrt{5}$. If $ab < 0$ and P is the point $(a - b, b, 2b - a)$ then CP^2 is equal to :

(1) $\frac{17}{3}$ (2) $\frac{16}{3}$

(3) $\frac{73}{3}$ (4) $\frac{97}{3}$

Official Ans. by NTA (1)

Ans. (1)

Sol. A(a, –2, 4), B(2, b, –3)

AC : CB = 2 : 1

$$\Rightarrow C \equiv \left(\frac{a+4}{3}, \frac{2b-2}{3}, \frac{-2}{3}\right)$$

C lies on $2x - y + 2 = 4$

$$\Rightarrow \frac{2a+8}{3} - \frac{2b-2}{3} - \frac{2}{3} = 4$$

$\Rightarrow a - b = 2 \ldots (1)$

Also OC = $\sqrt{5}$

$$\Rightarrow \left(\frac{a+4}{3}\right)^2 + \left(\frac{2b-2}{3}\right)^2 + \frac{4}{9} = 5 \quad \ldots (2)$$

Solving, (1) and (2)

$(b + 6)^2 + (2b - 2)^2 = 41$

$\Rightarrow 5b^2 + 4b - 1 = 0$

$\Rightarrow b = -1 \text{ or } \frac{1}{5}$

$\Rightarrow a = 1 \text{ or } \frac{11}{5}$

But $ab < 0 \Rightarrow (a, b) = (1, -1)$

$$C \equiv \left(\frac{5}{3}, \frac{-4}{3}, \frac{-2}{3}\right), P \equiv (2, -1, -3)$$

$$CP^2 = \frac{1}{9} + \frac{1}{9} + \frac{49}{9} = \frac{51}{9} = \frac{17}{3}$$

73. Let $\vec{a}=4\hat{i}+3\hat{j}$ and $\vec{b}=3\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}$ is a vector such that $\vec{c}.\left(\vec{a}\times\vec{b}\right)+25=0$, $\vec{c}.\left(\hat{i}+\hat{j}+\hat{k}\right)=4$ and projection of $\vec{c}$ on $\vec{a}$ is 1, then the projection of $\vec{c}$ on $\vec{b}$ equals :

(1) $\frac{5}{\sqrt{2}}$

(2) $\frac{1}{5}$

(3) $\frac{1}{\sqrt{2}}$

(4) $\frac{3}{\sqrt{2}}$

Official Ans. by NTA (1)

Ans. (1)

Sol. $\vec{a}\times\vec{b}=15\hat{i}-20\hat{j}-25\hat{k}$

Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

$\Rightarrow \quad 15x-20y-25z+25=0$

$\Rightarrow \quad 3x-4y-5z=-5$

Also $x+y+z=4$

and $\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|}=1 \Rightarrow 4x+3y=5$

$\Rightarrow \quad \vec{c}=2\hat{i}-\hat{j}+3\hat{k}$

Projection of $\vec{c}$ or $\vec{b}$ $=\frac{25}{5\sqrt{2}}=\frac{5}{\sqrt{2}}$

74. If the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{1}$ and $\frac{x-a}{2}=\frac{y+2}{3}=\frac{z-3}{1}$ intersects at the point P, then the distance of the point P from the plane $z=a$ is :

(1) 16 (2) 28

(3) 10 (4) 22

Official Ans. by NTA (2)

Ans. (2)

Sol. Point on $L_1 \equiv (\lambda+1, 2\lambda+2, \lambda-3)$

Point on $L_2 \equiv (2\mu+a, 3\mu-2, \mu+3)$

$\lambda-3=\mu+3 \quad \Rightarrow \lambda=\mu+6 \quad \ldots(1)$

$2\lambda+2=3\mu-2 \quad \Rightarrow 2\lambda=3\mu-4 \quad \ldots(2)$

Solving, (1) and (2)

$\Rightarrow \quad \lambda=22 \;\&\; \mu=16$

$\Rightarrow \quad P\equiv(23, 46, 19)$

$\Rightarrow \quad a=-9$

Distance of P from $z=-9$ is 28

75. The value of the integral $\int_{1/2}^{2}\frac{\tan^{-1}x}{x}dx$ is equal to

(1) $\pi\log_e 2$ (2) $\frac{1}{2}\log_e 2$

(3) $\frac{\pi}{4}\log_e 2$ (4) $\frac{\pi}{2}\log_e 2$

Official Ans. by NTA (4)

Ans. (4)

Sol. $I=\int_{1/2}^{2}\frac{\tan^{-1}x}{x}dx \quad \ldots\ldots(i)$

Put $x=\frac{1}{t} \quad dx=-\frac{1}{t^2}dt$

$$I=-\int_{2}^{1/2}\frac{\tan^{-1}\frac{1}{t}}{\frac{1}{t}}\cdot\frac{1}{t^2}dt=-\int_{2}^{1/2}\frac{\tan^{-1}\frac{1}{t}}{t}dt$$

$$I=\int_{1/2}^{2}\frac{\cot^{-1}t}{t}dt=\int_{1/2}^{2}\frac{\cot^{-1}x}{x}dx \quad \ldots\ldots(ii)$$

Add both equation

$$2I=\int_{1/2}^{2}\frac{\tan^{-1}x+\cot^{-1}x}{x}dx=\frac{\pi}{2}\int_{1/2}^{2}\frac{dx}{x}=\frac{\pi}{2}(\ell n2)_{1/2}^{2}$$

$$=\frac{\pi}{2}\left(\ell n2-\ell n\frac{1}{2}\right)=\pi\ell n2$$

$$I=\frac{\pi}{2}\ell n2$$

76. If the tangent at a point P on the parabola $y^2=3x$ is parallel to the line $x+2y=1$ and the tangents at the points Q and R on the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ are perpendicular to the line $x-y=2$, then the area of the triangle PQR is:

(1) $\frac{9}{\sqrt{5}}$ (2) $5\sqrt{3}$

(3) $\frac{3}{2}\sqrt{5}$ (4) $3\sqrt{5}$

Official Ans. by NTA (4)

Ans. (4)

Sol. $y^2 = 3x$

Tangent $P(x_1, y_1)$ is parallel to $x + 2y = 1$

Then slope at $P = -\frac{1}{2}$

$$2y\frac{dy}{dx} = 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2y} = -\frac{1}{2}$$

$$\Rightarrow y_1 = -3$$

Coordinates of P(3, –3)

Similarly $Q\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{5}}\right)$, $R\left(-\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

Area of ΔPQR

$$= \frac{1}{2}\begin{vmatrix} 3 & -3 & 1 \\ \frac{4}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \\ -\frac{4}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3\left(\frac{2}{\sqrt{5}}\right) + 3\left(\frac{8}{\sqrt{5}}\right) + 0\right] = \frac{30}{2\sqrt{5}} = 3\sqrt{5}$$

77. Let y = y(x) be the solution of the differential equation $x\log_e x\frac{dy}{dx} + y = x^2\log_e x, (x > 1)$. If y(2) = 2, then y(e) is equal to

(1) $\frac{4+e^2}{4}$

(2) $\frac{1+e^2}{4}$

(3) $\frac{2+e^2}{2}$

(4) $\frac{1+e^2}{2}$

Official Ans. by NTA (1)

Ans. (1)

Sol. $x\log_e x\frac{dy}{dx} + y = x^2\log_e x, (x > 1)$.

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x\ln x} = x$$

Linear differential equation

$$\text{I.F.} = e^{\int \frac{1}{x\ln x}dx} = |\ln x|$$

∴ Solution of differential equation

$$y|\ln x| = \int x|\ln x|dx$$

$$= |\ln x|\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx$$

$$\Rightarrow y|\ln x| = |\ln x|\left(\frac{x^2}{2}\right) - \frac{x^2}{4} + c$$

For constant

$y(2) = 2 \Rightarrow c = 1$

So, $y(x) = \frac{x^2}{2} - \frac{x^2}{4|\ln x|} + \frac{1}{|\ln x|}$

Hence, $y(e) = \frac{e^2}{2} - \frac{e^2}{4} + 1 = 1 + \frac{e^2}{4}$

78. The number of 3 digit numbers, that are divisible by either 3 or 4 but not divisible by 48, is

(1) 472 (2) 432

(3) 507 (4) 400

Official Ans. by NTA (2)

Ans. (2)

Sol. Total 3 digit number = 900

Divisible by 3 = 300 (Using $\frac{900}{3} = 300$)

Divisible by 4 = 225 (Using $\frac{900}{4} = 225$)

Divisible by 3 & 4 = 108, ….

(Using $\frac{900}{12} = 75$)

Number divisible by either 3 or 4

= 300 + 2250 – 75 = 450

We have to remove divisible by 48,

144, 192, ….., 18 terms

Required number of numbers = 450 – 18 = 432

79. Let R be a relation defined on $\mathbb{N}$ as a R b is $2a + 3b$ is a multiple of 5, a, b $\in \mathbb{N}$. Then R is

(1) not reflexive

(2) transitive but not symmetric

(3) symmetric but not transitive

(4) an equivalence relation

Official Ans. by NTA (4)

Ans. (4)

Sol. a R a $\Rightarrow$ 5a is multiple it 5

So reflexive

a R b $\Rightarrow 2a + 3b = 5\alpha$,

Now b R a

$$2b + 3a = 2b + \left(\frac{5\alpha - 3b}{2}\right) \cdot 3$$

$$= \frac{15}{2}\alpha - \frac{5}{2}b = \frac{5}{2}(3\alpha - b)$$

$$= \frac{5}{2}(2a + 2b - 2\alpha)$$

$$= 5(a + b - \alpha)$$

Hence symmetric

a R b $\quad \Rightarrow \quad 2a + 3b = 5\alpha$.

b R c $\quad \Rightarrow \quad 2b + 3c = 5\beta$

Now $\quad 2a + 5b + 3c = 5(\alpha + \beta)$

$\Rightarrow 2a + 5b + 3c = 5(\alpha + \beta)$

$\Rightarrow 2a + 3c = 5(\alpha + \beta - b)$

$\Rightarrow$ a R c

Hence relation is equivalence relation.

80. Consider a function $f : \mathbb{N} \to \mathbb{R}$, satisfying

$f(1) + 2f(2) + 3f(3) + \ldots + xf(x) = x(x + 1) f(x); x \ge 2$

with f(1)=1. Then $\frac{1}{f(2022)} + \frac{1}{f(2028)}$ is equal to

(1) 8200

(2) 8000

(3) 8400

(4) 8100

Official Ans. by NTA (4)

Ans. (4)

Sol. Given for $x \ge 2$

$$f(1) + 2f(2) + \ldots + xf(x) = x(x + 1) f(x)$$

replace x by x + 1

$$\Rightarrow \quad x(x + 1) f(x) + (x + 1) f(x + 1) = (x + 1)(x + 2) f(x + 1)$$

$$\Rightarrow \quad \frac{x}{f(x+1)} + \frac{1}{f(x)} = \frac{(x+2)}{f(x)}$$

$$\Rightarrow \quad x f(x) = (x + 1) f(x + 1) = \frac{1}{2}, x \ge 2$$

$$f(2) = \frac{1}{4},\ f(3) = \frac{1}{6}$$

Now $f(2022) = \frac{1}{4044}$

$$f(2028) = \frac{1}{4056}$$

So, $\frac{1}{f(2022)} + \frac{1}{f(2028)} = 4044 + 4056 = 8100$

SECTION-B

81. The total number of 4-digit numbers whose greatest common divisor with 54 is 2, is _____.

Official Ans. by NTA (3000)

Ans. (3000)

Sol. N should be divisible by 2 but not by 3

N = (Numbers divisible by 2) – (Numbers divisible by 6)

$$N = \frac{9000}{2} - \frac{9000}{6} = 4500 - 1500 = 3000$$

82. A triangle is formed by the tangents at the point (2, 2) on the curves $y^2 = 2x$ and $x^2 + y^2 = 4x$, and the line $x + y + 2 = 0$. If r is the radius of its circumcircle, then r^2 is equal to _______.

Official Ans. by NTA (10)

Ans. (10)

Sol. $S_1 : y^2 = 2x$ $\quad\quad S_2 : x^2 + y^2 = 4x$

P(2,2) is common point on S_1 & S_2

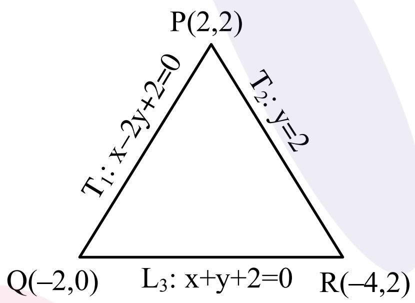
T_1 is tangent to S_1 at P $\Rightarrow T_1 : y.2 = x + 2$

$\Rightarrow T_1 : x - 2y + 2 = 0$

T_2 is tangent to S_2 at P $\Rightarrow T_2 : x.2 + y.2 = 2(x+2)$

$\Rightarrow T_2 : y = 2$

& $L_3 : x + y + 2 = 0$ is third line



$PQ = a = \sqrt{20}$

$QR = b = \sqrt{8}$

$RP = c = 6$

Area $(\Delta PQR) = \Delta = \frac{1}{2} \times 6 \times 2 = 6$

$\therefore\ r = \frac{abc}{4\Delta} = \frac{\sqrt{160}}{4} = \sqrt{10} \Rightarrow r^2 = 10$

83. A circle with centre (2, 3) and radius 4 intersects the line x + y = 3 at the points P and Q. If the tangents at P and Q intersect at the point $S(\alpha, \beta)$, then $4\alpha - 7\beta$ is equal to ________.

Official Ans. by NTA (11)

Ans. (11)

Sol. The given line is polar or $P(2, \beta)$ w.r.t. given circle

$x^2 + y^2 - 4x - 6y - 3 = 0$

Chord or contact

$\alpha x + \beta y - 2(x + \alpha) - 3(y + \beta) - 3 = 0$

$\Rightarrow (\alpha - 2)x + (\beta - 3)y - (2\alpha + 3\beta + 3) = 0$ ….. (i)

$\because$ But the equation of chord of contact is given

as : $x + y - 3 = 0$ …… (ii)

comparing the coefficients

$$\frac{\alpha - 2}{1} = \frac{\beta - 3}{1} = -\left(\frac{2\alpha + 3\beta + 3}{-3}\right)$$

On solving $\alpha = -6$

$\beta = -5$

Now $4\alpha - 7\beta = 11$

84. Let $a_1 = b_1 = 1$ and $a_n = a_{n-1} + (n - 1)$, $b_n = b_{n-1} + a_{n-1}$, $\forall\ n \geq 2$. If $S = \sum_{n=1}^{10} \frac{b_n}{2^n}$ and $T = \sum_{n=1}^{8} \frac{n}{2^{n-1}}$, then $2^7(2S - T)$ is equal to ______.

Official Ans. by NTA (461)

Ans. (461)

Sol. As, $S = \frac{b_1}{2} + \frac{b_2}{2^2} + \text{.......} + \frac{b_9}{2^9} + \frac{b_{10}}{2^{10}}$

$\Rightarrow \frac{S}{2} = \quad \frac{b_1}{2^2} + \frac{b_2}{2^3} + \text{.......} + \frac{b_9}{2^{10}} + \frac{b_{10}}{2^{11}}$

subtracting

$\Rightarrow \frac{S}{2} = \frac{b_1}{2} + \left(\frac{a_1}{2^2} + \frac{a_2}{2^3} \text{.......} + \frac{a_9}{2^{10}}\right) - \frac{b_{10}}{2^{11}}$

$\Rightarrow S = b_1 - \frac{b_{10}}{2^{10}} + \left(\frac{a_1}{2} + \frac{a_2}{2^2} \text{.......} + \frac{a_9}{2^9}\right)$

$\Rightarrow \frac{S}{2} = \frac{b_1}{2} - \frac{b_{10}}{2^{11}} + \left(\frac{a_1}{2^2} + \frac{a_2}{2^3} \text{.......} + \frac{a_9}{2^{10}}\right)$

subtracting

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} - \frac{b_{10}}{2^{11}} + \left(\frac{a_1}{2} - \frac{a_9}{2^{10}}\right) + \left(\frac{1}{2^2} + \frac{2}{2^3} + \ldots + \frac{8}{2^9}\right)$$

$$\Rightarrow \frac{S}{2} = \frac{a_1 + b_1}{2} - \frac{(b_{10} + 2a_9)}{2^{11}} + \frac{T}{4}$$

$$\Rightarrow 2S = 2(a_1 + b_1) - \frac{(b_{10} + 2a_9)}{2^9} + T$$

$$\Rightarrow 2^7 (2S - T) = 2^8 (a_1 + b_1) - \frac{(b_{10} + 2a_9)}{4}$$

Given $a_n - a_{n-1} = n - 1,$

$\therefore$ $a_2 - a_1 = 1$

$a_3 - a_2 = 2$

$\vdots$

$a_9 - a_8 = 8$

$a_9 - a_1 = 1 + 2 + \ldots + 8 = 36$

$\Rightarrow$ $a_9 = 37 \quad (a_1 = 1)$

Also, $b_n - b_{n-1} = a_{n-1}$

$\therefore$ $b_{10} - b_1 = a_1 + a_2 + \ldots + a_9$

$= 1 + 2 + 4 + 7 + 11 + 16 + 22 + 29 + 37$

$\Rightarrow$ $b_{10} = 130 \quad (\text{As } b_1 = 1)$

$\therefore$ $2^7 (2S - T) = 2^8 (1 + 1) - (130 + 2 \times 37)$

$$2^9 - \frac{204}{4} = 461$$

85. If the equation of the normal to the curve $y = \frac{x - a}{(x + b)(x - 2)}$ at the point $(1, -3)$ is $x - 4y = 13$, then the value of $a + b$ is equal to _____.

Official Ans. by NTA (4)

Ans. (4)

Sol. $y = \frac{x - a}{(x + b)(x - 2)}$

At point $(1, -3)$,

$$-3 = \frac{1 - 9}{(1 + b)(1 - 2)}$$

$\Rightarrow$ $1 - a = 3(1 + b)$ ….. (1)

Now, $y = \frac{x - a}{(x + b)(x - 2)}$

$$\Rightarrow \frac{dy}{dx} = \frac{(x + b)(x - 2) \times (1) - (x - a)(2x + b - 2)}{(x + b)^2 (x - 2)^2}$$

At $(1, -3)$ slope of normal is $\frac{1}{4}$ hence $\frac{dy}{dx} = -4,$

So, $-4 = \frac{(1 + b)(-1) - (1 - a)b}{(1 + b)^2 (-1)^2}$

Using equation (1)

$$\Rightarrow -4 = \frac{(1 + b)(-1) - 3(b + 1)b}{(1 + b)^2}$$

$$\Rightarrow -4 = \frac{(-1) - 3b}{(1 + b)} (b \neq -1)$$

$\Rightarrow$ $b = -3$

So, $a = 7$

Hence, $a + b = 7 - 3 = 4$

86. Let A be a symmetric matrix such that $|A| = 2$ and $\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$. If the sum of the diagonal elements of A is s, then $\frac{\beta s}{\alpha^2}$ is equal to _____.

Official Ans. by NTA (5)

Ans. (5)

Sol. $\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$

Now $ac - b^2 = 2$ and $2a + b = 1$

and $2b + c = 2$

solving all these above equations we get

$$\frac{1 - b}{2} \times \left(\frac{2 - 2b}{1}\right) - b^2 = 2$$

$\Rightarrow$ $(1 - b)^2 - b^2 = 2$

$\Rightarrow$ $1 - 2b = 2$

$\Rightarrow$ $b = -\frac{1}{2}$ and $a = \frac{3}{4}$ and $c = 3$

Hence $\alpha = 3a + \frac{3b}{2} = \frac{9}{4} - \frac{3}{4} = \frac{3}{2}$

and $\beta = 3b + \frac{3c}{2} = -\frac{3}{2} + \frac{9}{2} = 3$

also $s = a + c = \frac{15}{4}$

$$\therefore \frac{\beta s}{\alpha^2} = \frac{3 \times 15}{4 \times \frac{9}{4}} = 5$$

87. Let $\{a_k\}$ and $\{b_k\}$, $k \in \mathbb{N}$, be two G.P.s with common ratio r_1 and r_2 respectively such that $a_1 = b_1 = 4$ and $r_1 < r_2$. Let $c_k = a_k + b_k$, $k \in \mathbb{N}$. If $c_2 = 5$ and $c_3 = \frac{13}{4}$ then $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$ is equal to _______.

Official Ans. by NTA (9)

Ans. (9)

Sol. Given that

$c_k = a_k + b_k$ and $a_1 = b_1 = 4$

also $a_2 = 4r_1$ $\quad a_3 = 4r_1^2$

$b_2 = 4r_2$ $\quad b_3 = 4r_2^2$

Now $c_2 = a_2 + b_2 = 5$ and $c_3 = a_3 + b_3 = \frac{13}{4}$

$$\Rightarrow r_1 + r_2 = \frac{5}{4} \text{ and } r_1^2 + r_2^2 = \frac{13}{16}$$

Hence $r_1 r_2 = \frac{3}{8}$ which gives $r_1 = \frac{1}{2}$ & $r_2 = \frac{3}{4}$

$$\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$$

$$= \frac{4}{1-r_1} + \frac{4}{1-r_2} - \left(\frac{48}{32} + \frac{27}{2}\right)$$

$$= 24 - 15 = 9$$

88. Let X = {11, 12, 13, ….., 40, 41} and Y = {61, 62, 63, ….., 90, 91} be the two sets of observations. If $\bar{x}$ and $\bar{y}$ are their respective means and σ^2 is the variance of all the observations in $X \cup Y$, then $\left|\bar{x} + \bar{y} - \sigma^2\right|$ is equal to _______.

Official Ans. by NTA (603)

Ans. (603)

Sol. $\bar{x} = \frac{\sum_{i=11}^{41} i}{31} = \frac{11+41}{2} = 26$ (31 elements)

$\bar{y} = \frac{\sum_{j=61}^{91} j}{31} = \frac{61+91}{2} = 76$ (31 elements)

Combined mean, $\mu = \frac{31\times 26 + 31\times 76}{31+31}$

$$= \frac{26+76}{2} = 51$$

$$\sigma^2 = \frac{1}{62} \times \left(\sum_{i=1}^{31} (x_i - \mu)^2 + \sum_{i=1}^{31} (y_i - \mu)^2\right) = 705$$

Since, $x_i \in X$ are in A.P. with 31 elements & common difference 1, same is $y_i \in y$, when written in increasing order.

$$\therefore \sum_{i=1}^{31} (x_i - \mu)^2 = \sum_{i=1}^{31} (y_i - \mu)^2$$

$$= 10^2 + 11^2 + \ldots.. + 40^2$$

$$= \frac{40\times 41\times 81}{6} - \frac{9\times 10\times 19}{6} = 21855$$

$$\therefore \left|\bar{x} + \bar{y} - \sigma^2\right| = \left|26+76-705\right| = 603$$

89. Let $\alpha = 8 - 14i$, $A = \left\{z \in \mathbb{C} : \frac{\alpha z - \bar{\alpha}\bar{z}}{z^2 - (\bar{z})^2 - 112i} = 1\right\}$ and $B = \{z \in \mathbb{C} : |z + 3i| = 4\}$.

Then $\sum_{z \in A \cap B} (\text{Re}\, z - \text{Im}\, z)$ is equal to _____.

Official Ans. by NTA (14)

Ans. (14)

Sol. $\alpha = 8 - 14i$

$z = x + iy$

$az = (8x + 14y) + i(-14x + 8y)$

$z+\bar{z}=2x \quad z-\bar{z}=2iy$

Set A: $\dfrac{2i(-14x+8y)}{i(4xy-112)}=1$

$(x-4)(y+7)=0$

$x=4$ or $y=-7$

Set B: $x^2+(y+3)^2=16$

when $x=4$ $\quad y=-3$

when $y=-7$ $\quad x=0$

$\therefore A\cap B=\{4-3i, 0-7i\}$

So, $\sum\limits_{z\in A\cap B}(\text{Re}\,z-\text{Im}\,z) = 4-(-3)+(0-(-7))=14$

90. Let $\alpha_1, \alpha_2, \ldots, \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_7|$. Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to______.

Official Ans. by NTA (9)

Ans. (9)

Sol. Given equation can be rearranged as

$x(x^6+3x^4-13x^2-15)=0$

clearly $x=0$ is one of the root and other part can be observed by replacing $x^2 = t$ from which we have $\quad t^3+3t^2-13t-15=0$

$\Rightarrow \quad (t-3)(t^2+6t+5)=0$

So, $\quad t=3,\ t=-1,\ t=-5$

Now we are getting $x^2=3, x^2=-1, x^2=-5$

$\Rightarrow\ x=\pm\sqrt{3},\ x=\pm i,\ x=\pm\sqrt{5}i$

From the given condition $|\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_7|$

We can clearly say that $|\alpha_7|=0$ and

and $\quad |\alpha_6| = \sqrt{5} = |\alpha_5|$

and $\quad |\alpha_4| = \sqrt{3} = |\alpha_3|$ and $|\alpha_2| = 1 = |\alpha_1|$

So we can have, $\alpha_1 = \sqrt{5}i$, $\alpha_2 = -\sqrt{5}i$, $\alpha_3 = \sqrt{3}i$,

$\alpha_4 = -\sqrt{3}$, $\alpha_5 = i$, $\alpha_6 = -i$

Hence

$\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$

$= 1-(-3)+5=9$