Exercise 5.2

Question 1:

Differentiate the functions with respect to *x*.

$\sin\left(x^2+5\right)$

Answer

Let $f(x)=\sin\left(x^2+5\right),\ u(x)=x^2+5,$ and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, *f* is a composite of two functions.

Put $t=u(x)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

Therefore, by chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right)\cdot\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right)\cdot[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect to *x*.

$\cos(\sin x)$

Answer

$$\text{Let } f(x)=\cos(\sin x), u(x)=\sin x, \text{ and } v(t)=\cos t$$

$$\text{Then, } (vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$$

Thus, *f* is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}\left[\cos(\sin x)\right]=-\sin(\sin x)\cdot\frac{d}{dx}(\sin x)=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$$

Question 3:

Differentiate the functions with respect to *x*.

$$\sin(ax+b)$$

Answer

$$\text{Let } f(x)=\sin(ax+b),\ u(x)=ax+b, \text{ and } v(t)=\sin t$$

$$\text{Then, } (vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$$

Thus, *f* is a composite function of two functions, *u* and *v*.

Put $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=\cos(ax+b).(a+0)$$
$$=a\cos(ax+b)$$

Question 4:

Differentiate the functions with respect to *x*.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Answer

Let $f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t,$ and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, *f* is a composite function of three functions, *u, v*, and *w*.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

Then, $\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t)$ $\quad [s=\tan t]$

$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)$ $\quad [t=\sqrt{x}]$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to *x*.

$$\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}$$

Answer

The given function is $f\left(x\right)=\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}=\frac{g\left(x\right)}{h\left(x\right)}$, where $g(x) = \sin(ax + b)$ and $h(x) = \cos(cx + d)$

$$\therefore f'=\frac{g'h-gh'}{h^2}$$

Consider $g\left(x\right)=\sin\left(ax+b\right)$

Let $u\left(x\right)=ax+b, v\left(t\right)=\sin t$

Then, $\left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)=v\left(ax+b\right)=\sin\left(ax+b\right)=g\left(x\right)$

$\therefore$ g is a composite function of two functions, u and v.

Put $t = u(x) = ax + b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx} = \cos(ax+b)\cdot a = a\cos(ax+b)$$

Consider $h(x) = \cos(cx+d)$

Let $p(x) = cx + d,\ q(y) = \cos y$

Then, $(qop)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$

$\therefore h$ is a composite function of two functions, p and q.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx+d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx+d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy}\cdot\frac{dy}{dx} = -\sin(cx+d)\times c = -c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to *x*.

$\cos x^3.\sin^2(x^5)$

Answer

The given function is $\cos x^3.\sin^2(x^5)$.

$$\frac{d}{dx}\left[\cos x^3\cdot\sin^2(x^5)\right] = \sin^2(x^5)\times\frac{d}{dx}(\cos x^3) + \cos x^3\times\frac{d}{dx}\left[\sin^2(x^5)\right]$$

$$= \sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3) + \cos x^3\times 2\sin(x^5)\cdot\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3\sin^2(x^5)\times 3x^2 + 2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}(x^5)$$

$$= -3x^2\sin x^3\cdot\sin^2(x^5) + 2\sin x^5\cos x^5\cos x^3\cdot\times 5x^4$$

$$= 10x^4\sin x^5\cos x^5\cos x^3 - 3x^2\sin x^3\sin^2(x^5)$$

Question 7:

Differentiate the functions with respect to *x*.

$2\sqrt{\cot(x^2)}$

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$
$$= 2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$
$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times -\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$
$$= -\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times(2x)$$
$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$
$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\sin x^2}$$
$$= \frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the functions with respect to *x*.

$\cos\left(\sqrt{x}\right)$

Answer

Let $f\left(x\right) = \cos\left(\sqrt{x}\right)$

Also, let $u\left(x\right) = \sqrt{x}$

And, $v\left(t\right) = \cos t$

Then, $\left(vou\right)\left(x\right) = v\left(u\left(x\right)\right)$

$$= v\left(\sqrt{x}\right)$$
$$= \cos\sqrt{x}$$
$$= f\left(x\right)$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

$t = u\left(x\right) = \sqrt{x}$

Then, $\frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{x}}$$

And, $\frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t$

$$= -\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\frac{dt}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

$$= -\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] = -\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= -\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function *f* given by

$f(x) = |x-1|,\ x \in \mathbf{R}$ is notdifferentiable at x = 1.

Answer

The given function is $f(x) = |x-1|,\ x \in \mathbf{R}$

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at *x* = 1,

consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^-} \frac{|h|-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} \qquad (h<0 \Rightarrow |h| = -h)$$

$$= -1$$

Consider the right hand limit of f at $x = 1$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} \qquad (h>0 \Rightarrow |h| = h)$$

$$= 1$$

Since the left and right hand limits of *f* at *x* = 1 are not equal, *f* is not differentiable at *x* = 1

Question 10:

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at *x* = 1 and *x* = 2.

Answer

The given function *f* is $f(x) = [x], 0 < x < 3$

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at *x* = 1, consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiability of the given function at x = 2, consider the left hand limit of f at x = 2

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{2-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at x = 2