Exercise 1.3

Question 1:

Let *f*: {1, 3, 4} → {1, 2, 5} and *g*: {1, 2, 5} → {1, 3} be given by *f* = {(1, 2), (3, 5), (4, 1)} and *g* = {(1, 3), (2, 3), (5, 1)}. Write down *gof*.

Answer

The functions *f*: {1, 3, 4} → {1, 2, 5} and *g*: {1, 2, 5} → {1, 3} are defined as *f* = {(1, 2), (3, 5), (4, 1)} and *g* = {(1, 3), (2, 3), (5, 1)}.

$gof(1) = g(f(1)) = g(2) = 3 \qquad [f(1) = 2 \text{ and } g(2) = 3]$

$gof(3) = g(f(3)) = g(5) = 1 \qquad [f(3) = 5 \text{ and } g(5) = 1]$

$gof(4) = g(f(4)) = g(1) = 3 \qquad [f(4) = 1 \text{ and } g(1) = 3]$

$\therefore gof = \{(1,3),(3,1),(4,3)\}$

Question 2:

Let *f*, *g* and *h* be functions from **R** to **R**. Show that

$(f+g)oh = foh + goh$

$(f \cdot g)oh = (foh)\cdot(goh)$

Answer

To prove:

$(f+g)oh = foh + goh$

Consider:

$((f+g)oh)(x)$

$= (f+g)(h(x))$

$= f(h(x)) + g(h(x))$

$= (foh)(x) + (goh)(x)$

$= \{(foh)+(goh)\}(x)$

$\therefore ((f+g)oh)(x) = \{(foh)+(goh)\}(x) \qquad \forall x \in \mathbf{R}$

Hence, $(f+g)oh = foh + goh$.

To prove:

$(f \cdot g)oh = (foh)\cdot(goh)$

Consider:

$((f \cdot g)oh)(x)$

$= (f \cdot g)(h(x))$

$= f(h(x)).g(h(x))$

$= (foh)(x).(goh)(x)$

$= \{(foh).(goh)\}(x)$

$\therefore ((f \cdot g)oh)(x) = \{(foh).(goh)\}(x)\ \forall x \in \mathrm{R}$

Hence, $(f \cdot g)oh = (foh)\cdot(goh)$.

Question 3:

Find *gof* and *fog*, if

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Answer

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

$\therefore (gof)(x) = g(f(x)) = g(|x|) = |5|x|-2|$

$(fog)(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$\therefore (gof)(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$

$(fog)(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$

Question 4:

If $f(x)=\frac{(4x+3)}{(6x-4)}, x\neq\frac{2}{3}$, show that *f* o *f*(*x*) = *x*, for all $x\neq\frac{2}{3}$. What is the inverse of *f*?

Answer

It is given that $f(x)=\frac{(4x+3)}{(6x-4)}, x\neq\frac{2}{3}$.

$$(fof)(x)=f(f(x))=f\left(\frac{4x+3}{6x-4}\right)$$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x$$

$$\text{Therefore, } fof(x)=x, \text{ for all } x\neq\frac{2}{3}.$$

$$\Rightarrow fof=\text{I}$$

Hence, the given function *f* is invertible and the inverse of *f* is *f* itself.

Question 5:

State with reason whether following functions have inverse

(i) *f*: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) *g*: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) *h*: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Answer

(i) *f*: {1, 2, 3, 4} → {10}defined as:

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of *f*, we can see that *f* is a many one function as: *f*(1) = *f*(2) = *f*(3) = *f*(4) = 10

∴*f* is not one-one.

Hence, function *f* does not have an inverse.

(ii) *g*: {5, 6, 7, 8} → {1, 2, 3, 4} defined as:

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

From the given definition of *g*, it is seen that *g* is a many one function as: *g*(5) = *g*(7) = 4.

∴*g* is not one-one,

Hence, function g does not have an inverse.

(iii) *h*: {2, 3, 4, 5} → {7, 9, 11, 13} defined as:

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under *h*.

∴Function *h* is one-one.

Also, *h* is onto since for every element *y* of the set {7, 9, 11, 13}, there exists an element *x* in the set {2, 3, 4, 5}such that *h*(*x*) = *y*.

Thus, *h* is a one-one and onto function. Hence, *h* has an inverse.

Question 6:

Show that *f*: [−1, 1] → **R**, given by $f(x)=\frac{x}{(x+2)}$ is one-one. Find the inverse of the function *f*: [−1, 1] → Range *f*.

(Hint: For *y* ∈Range *f*, *y* = $f(x)=\frac{x}{x+2}$, for some *x* in [−1, 1], i.e., $x=\frac{2y}{(1-y)}$)

Answer

f: [−1, 1] → R is given as $f(x)=\frac{x}{(x+2)}$.

Let *f*(*x*) = *f*(*y*).

$$\Rightarrow \frac{x}{x+2}=\frac{y}{y+2}$$

$$\Rightarrow xy+2x=xy+2y$$

$$\Rightarrow 2x=2y$$

$$\Rightarrow x=y$$

∴ *f* is a one-one function.

It is clear that *f*: [−1, 1] → Range *f* is onto.

∴ *f*: [−1, 1] → Range *f* is one-one and onto and therefore, the inverse of the function:

f: [−1, 1] → Range *f* exists.

Let *g*: Range *f* → [−1, 1] be the inverse of *f*.

Let *y* be an arbitrary element of range *f*.

Since *f*: [−1, 1] → Range *f* is onto, we have:

$$y = f(x) \text{ for same } x \in [-1,\ 1]$$

$$\Rightarrow y = \frac{x}{x+2}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y},\ y \neq 1$$

Now, let us define *g*: Range *f* → [−1, 1] as

$$g(y) = \frac{2y}{1-y},\ y \neq 1.$$

$$\text{Now, } (gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

∴*gof* = $I_{[-1,\ 1]}$ and *fog* = $I_{\text{Range} f}$

∴ $f^{-1} = g$

$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$$

Question 7:

Consider *f*: **R** → **R** given by *f*(*x*) = 4*x* + 3. Show that *f* is invertible. Find the inverse of *f*.

Answer

f: **R** → **R** is given by,

f(*x*) = 4*x* + 3

One-one:

Let $f(x) = f(y)$.

$\Rightarrow 4x+3=4y+3$

$\Rightarrow 4x=4y$

$\Rightarrow x=y$

∴ f is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$\Rightarrow x=\frac{y-3}{4} \in \mathbf{R}$

Therefore, for any $y \in \mathbf{R}$, there exists $x=\frac{y-3}{4} \in \mathbf{R}$ such that

$f(x)=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y.$

∴ f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $\mathbf{R} \rightarrow \mathbf{R}$ by $g(x)=\frac{y-3}{4}$.

$\text{Now,}(gof)(x)=g(f(x))=g(4x+3)=\frac{(4x+3)-3}{4}=x$

$(fog)(y)=f(g(y))=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y-3+3=y$

∴ $gof=fog=I_R$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y)=g(y)=\frac{y-3}{4}.$

Question 8:

Consider f: $\mathbf{R}_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y)=\sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

Answer

f: $\mathbf{R}_+ \rightarrow [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one:

Let $f(x) = f(y)$.

$$\Rightarrow x^2 + 4 = y^2 + 4$$
$$\Rightarrow x^2 = y^2$$
$$\Rightarrow x = y \qquad [\text{as } x = y \in \mathbf{R}_+]$$

$\therefore$ *f* is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$$\Rightarrow x^2 = y - 4 \geq 0 \qquad [\text{as } y \geq 4]$$
$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \sqrt{y-4} \in \mathbf{R}$ such that

$$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y.$$

$\therefore$ *f* is onto.

Thus, *f* is one-one and onto and therefore, f^{-1} exists.

Let us define $g: [4, \infty) \to \mathbf{R}_+$ by,

$$g(y) = \sqrt{y-4}$$

$$\text{Now, } gof(x) = g(f(x)) = g(x^2+4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$

$$\text{And, } fog(y) = f(g(y)) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$$

$$\therefore gof = fog = I_{\mathbf{R}+}$$

Hence, *f* is invertible and the inverse of *f* is given by

$$f^{-1}(y) = g(y) = \sqrt{y-4}.$$

Question 21:

Find the values of $\tan^{-1}\sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right)$ is equal to

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$

Answer

Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$ where $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let $\cot^{-1}\left(-\sqrt{3}\right) = y$.

Then, $\cot y = -\sqrt{3} = -\cot\left(\frac{\pi}{6}\right) = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6}$ where $\frac{5\pi}{6} \in (0, \pi)$.

The range of the principal value branch of $\cot^{-1}$ is $(0, \pi)$.

$\therefore \cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$

$\therefore \tan^{-1}\sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$

The correct answer is B.

Question 9:

Consider f: $\mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right) - 1}{3}\right)$.

Answer

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow (3x+1)^2 = y + 6$

$\Rightarrow 3x + 1 = \sqrt{y+6} \qquad [\text{as } y \geq -5 \Rightarrow y + 6 > 0]$

$\Rightarrow x = \frac{\sqrt{y+6} - 1}{3}$

$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \to \mathbf{R}_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$.

We now have:

$$(gof)(x) = g(f(x)) = g(9x^2 + 6x - 5)$$
$$= g\left((3x+1)^2 - 6\right)$$
$$= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3}$$
$$= \frac{3x+1-1}{3} = x$$

$$\text{And,} (fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right)$$
$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$$
$$= \left(\sqrt{y+6}\right)^2 - 6 = y + 6 - 6 = y$$

$\therefore gof = I_{\mathbf{R}_+}$ and $fog = I_{[-5, \infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Question 10:

Let f: $X \to Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$,

$fog_1(y) = I_Y(y) = fog_2(y)$. Use one-one ness of f).

Answer

Let f: $X \to Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have:

$fog_1(y) = I_Y(y) = fog_2(y)$

$\Rightarrow f(g_1(y)) = f(g_2(y))$

$\Rightarrow g_1(y) = g_2(y)$ $\quad [f \text{ is invertible} \Rightarrow f \text{ is one-one}]$

$\Rightarrow g_1 = g_2$ $\quad [g \text{ is one-one}]$

Hence, *f* has a unique inverse.

Question 11:

Consider *f*: {1, 2, 3} → {*a*, *b*, *c*} given by *f*(1) = *a*, *f*(2) = *b* and *f*(3) = *c*. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer

Function *f*: {1, 2, 3} → {*a*, *b*, *c*} is given by,

f(1) = *a*, *f*(2) = *b,* and *f*(3) = *c*

If we define *g*: {*a*, *b*, *c*} → {1, 2, 3} as *g*(*a*) = 1, *g*(*b*) = 2, *g*(*c*) = 3, then we have:

$(fog)(a) = f(g(a)) = f(1) = a$

$(fog)(b) = f(g(b)) = f(2) = b$

$(fog)(c) = f(g(c)) = f(3) = c$

And,

$(gof)(1) = g(f(1)) = g(a) = 1$

$(gof)(2) = g(f(2)) = g(b) = 2$

$(gof)(3) = g(f(3)) = g(c) = 3$

$\therefore gof = I_X$ and $fog = I_Y$, where *X* = {1, 2, 3} and *Y*= {*a*, *b*, *c*}.

Thus, the inverse of *f* exists and f^{-1} **=** *g*.

$\therefore f^{-1}$: {*a*, *b*, *c*} → {1, 2, 3} is given by,

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of *g*.

If we define *h*: {1, 2, 3} → {*a*, *b*, *c*} as

h(1) = *a*, *h*(2) = *b*, *h*(3) = *c*, then we have:

$(g \circ h)(1) = g(h(1)) = g(a) = 1$

$(g \circ h)(2) = g(h(2)) = g(b) = 2$

$(g \circ h)(3) = g(h(3)) = g(c) = 3$

And,

$(h \circ g)(a) = h(g(a)) = h(1) = a$

$(h \circ g)(b) = h(g(b)) = h(2) = b$

$(h \circ g)(c) = h(g(c)) = h(3) = c$

$\therefore\ goh = I_X$ and $hog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

Question 12:

Let f: $X \rightarrow Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

Answer

Let f: $X \rightarrow Y$ be an invertible function.

Then, there exists a function g: $Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$.

Here, $f^{-1} = g$.

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, f^{-1}: $Y \rightarrow X$ is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

Question 13:

If f: **R** → **R** be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Answer

f: **R** → **R** is given as $f(x)=\left(3-x^3\right)^{\frac{1}{3}}$.

$$f(x)=\left(3-x^3\right)^{\frac{1}{3}}$$

$$\therefore fof(x)=f(f(x))=f\left(\left(3-x^3\right)^{\frac{1}{3}}\right)=\left[3-\left(\left(3-x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$=\left[3-\left(3-x^3\right)\right]^{\frac{1}{3}}=\left(x^3\right)^{\frac{1}{3}}=x$$

$$\therefore fof(x)=x$$

The correct answer is C.

Question 14:

Let $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ be a function defined as $f(x)=\frac{4x}{3x+4}$. The inverse of *f* is map *g*: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ given by

(A) $g(y)=\frac{3y}{3-4y}$ (B) $g(y)=\frac{4y}{4-3y}$

(C) $g(y)=\frac{4y}{3-4y}$ (D) $g(y)=\frac{3y}{4-3y}$

Answer

It is given that $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ is defined as $f(x)=\frac{4x}{3x+4}$.

Let *y* be an arbitrary element of Range *f*.

Then, there exists *x* ∈ $\mathbf{R}-\left\{-\frac{4}{3}\right\}$ such that $y=f(x)$.

$$\Rightarrow y=\frac{4x}{3x+4}$$
$$\Rightarrow 3xy+4y=4x$$
$$\Rightarrow x(4-3y)=4y$$
$$\Rightarrow x=\frac{4y}{4-3y}$$

Let us define *g*: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ as $g(y)=\frac{4y}{4-3y}$.

Now, $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{4x}{3x+4}\right)$

$$= \frac{4\left(\dfrac{4x}{3x+4}\right)}{4-3\left(\dfrac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

$$\text{And,} (f \circ g)(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\dfrac{4y}{4-3y}\right)}{3\left(\dfrac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

$\therefore$ $gof = \mathrm{I}_{\mathbf{R}-\left\{-\frac{4}{3}\right\}}$ and $fog = \mathrm{I}_{\text{Range}\, f}$

Thus, *g* is the inverse of *f* i.e., $f^{-1} = g$.

Hence, the inverse of *f* is the map *g*: Range $f \to \mathbf{R} - \left\{-\dfrac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct answer is B.