Chapter 6

APPLICATION OF DERIVATIVES

6.1 Overview

6.1.1 *Rate of change of quantities*

For the function $y = f(x)$, $\frac{d}{dx}(f(x))$ represents the rate of change of y with respect to x.

Thus if 's' represents the distance and 't' the time, then $\frac{ds}{dt}$ represents the rate of change of distance with respect to time.

6.1.2 *Tangents and normals*

A line touching a curve $y = f(x)$ at a point (x_1, y_1) is called the tangent to the curve at that point and its equation is given $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$.

The normal to the curve is the line perpendicular to the tangent at the point of contact, and its equation is given as:

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

The angle of intersection between two curves is the angle between the tangents to the curves at the point of intersection.

6.1.3 *Approximation*

Since $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$, we can say that $f'(x)$ is approximately equal to $\frac{f(x + \Delta x) - f(x)}{\Delta x}$

$\Rightarrow$ approximate value of $f(x + \Delta x) = f(x) + \Delta x . f'(x)$.

6.1.4 *Increasing/decreasing functions*

A continuous function in an interval (a, b) is :

(i) strictly increasing if for all $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or for all $x \in (a, b)$, $f'(x) > 0$

(ii) strictly decreasing if for all $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or for all $x \in (a, b)$, $f'(x) < 0$

6.1.5 *Theorem* : Let f be a continuous function on $[a, b]$ and differentiable in (a, b) then

(i) f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(ii) f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(iii) f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

6.1.6 *Maxima and minima*

Local Maximum/Local Minimum for a real valued function *f*

A point c in the interior of the domain of f, is called

(i) local maxima, if there exists an $h > 0$, such that $f(c) > f(x)$, for all x in $(c - h, c + h)$.

The value $f(c)$ is called the local maximum value of f.

(ii) local minima if there exists an $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$.

The value $f(c)$ is called the local minimum value of f.

A function f defined over $[a, b]$ is said to have maximum (or absolute maximum) at $x = c$, $c \in [a, b]$, if $f(x) \le f(c)$ for all $x \in [a, b]$.

Similarly, a function $f(x)$ defined over $[a, b]$ is said to have a minimum [or absolute minimum] at $x = d$, if $f(x) \ge f(d)$ for all $x \in [a, b]$.

6.1.7 *Critical point of f* : A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

Working rule for finding points of local maxima or local minima:

(a) **First derivative test:**

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, then c is a point of local maxima, and $f(c)$ is local maximum value.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, then c is a point of local minima, and $f(c)$ is local minimum value.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local minima nor a point of local maxima. Such a point is called a point of inflection.

(b) **Second Derivative test:** Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. In this case $f(c)$ is then the local maximum value.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. In this case $f(c)$ is the local minimum value.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to first derivative test.

6.1.8 *Working rule for finding absolute maxima and or absolute minima :*

Step 1 : Find all the critical points of f in the given interval.

Step 2 : At all these points and at the end points of the interval, calculate the values of f.

Step 3 : Identify the maximum and minimum values of f out of the values calculated in step 2. The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

6.2 Solved Examples

Short Answer Type (S.A.)

Example 1 For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$?

Solution Slope of curve $= \dfrac{dy}{dx} = 5 - 6x^2$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x.\frac{dx}{dt}$$

$$= -12 \, . \, (3) \, . \, (2)$$

$$= -72 \text{ units/sec.}$$

Thus, slope of curve is decreasing at the rate of 72 units/sec when x is increasing at the rate of 2 units/sec.

Example 2 Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/sec in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, find the rate of decrease of the slant height of water.

Solution If s represents the surface area, then

$$\frac{ds}{dt} = 2\text{cm}^2 \text{ /sec}$$

$$s = \pi \, r.l = \pi l \, . \, \sin \frac{\pi}{4} \cdot l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore, $\frac{ds}{dt} = \frac{2\pi}{\sqrt{2}} l . \frac{dl}{dt} = \sqrt{2}\pi l . \frac{dl}{dt}$

when l = 4 cm, $\frac{dl}{dt} = \frac{1}{\sqrt{2}\pi.4} .2 = \frac{1}{2\sqrt{2}\pi} = \frac{\sqrt{2}}{4\pi}$ cm/s .

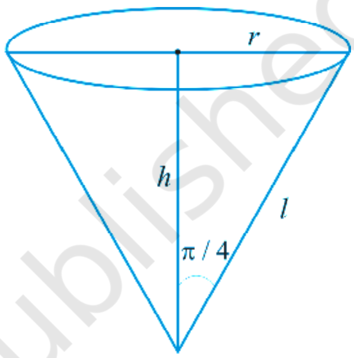

Fig. 6.1

Example 3 Find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$.

Solution Solving the given equations, we have $y^2 = x$ and $x^2 = y \Rightarrow x^4 = x$ or $x^4 - x = 0$

$\Rightarrow x \, (x^3 - 1) = 0 \Rightarrow x = 0, x = 1$

Therefore, $y = 0, \; y = 1$

i.e. points of intersection are (0, 0) and (1, 1)

Further $y^2 = x \Rightarrow 2y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

and $x^2 = y \Rightarrow \frac{dy}{dx} = 2x.$

At (0, 0), the slope of the tangent to the curve $y^2 = x$ is parallel to y-axis and the tangent to the curve $x^2 = y$ is parallel to x-axis.

$\Rightarrow$ angle of intersection = $\frac{\pi}{2}$

At (1, 1), slope of the tangent to the curve $y_2 = x$ is equal to $\frac{1}{2}$ and that of $x^2 = y$ is 2.

$$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+1}\right| = \frac{3}{4}. \qquad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 4 Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$.

Solution $f(x) = \tan x - 4x \Rightarrow f'(x) = \sec^2 x - 4$

When $\frac{-\pi}{3} < x < \frac{\pi}{3}$, $1 < \sec x < 2$

Therefore, $1 < \sec^2 x < 4 \Rightarrow -3 < (\sec^2 x - 4) < 0$

Thus for $\frac{-\pi}{3} < x < \frac{\pi}{3}$, $f'(x) < 0$

Hence f is strictly decreasing on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$.

Example 5 Determine for which values of x, the function $y = x^4 - \frac{4x^3}{3}$ is increasing and for which values, it is decreasing.

Solution $y = x^4 - \frac{4x^3}{3}$ $\qquad \Rightarrow \frac{dy}{dx} = 4x^3 - 4x^2 = 4x^2(x-1)$

Now, $\frac{dy}{dx} = 0 \Rightarrow x = 0, x = 1$.

Since $f'(x) < 0 \ \forall x \in (-\infty, 0) \cup (0, 1)$ and f is continuous in $(-\infty, 0]$ and $[0, 1]$. Therefore f is decreasing in $(-\infty, 1]$ and f is increasing in $[1, \infty)$.

Note: Here f is strictly decreasing in $(-\infty, 0) \cup (0, 1)$ and is strictly increasing in $(1, \infty)$.

Example 6 Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Solution $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27 = 3(4x^2 - 12x + 9) = 3(2x - 3)^2$

$f'(x) = 0 \Rightarrow x = \frac{3}{2}$ (critical point)

Since $f'(x) > 0$ for all $x < \frac{3}{2}$ and for all $x > \frac{3}{2}$

Hence $x = \frac{3}{2}$ is a point of inflexion i.e., neither a point of maxima nor a point of minima.

$x = \frac{3}{2}$ is the only critical point, and f has neither maxima nor minima.

Example 7 Using differentials, find the approximate value of $\sqrt{0.082}$

Solution Let $f(x) = \sqrt{x}$

Using $f(x + \Delta x) \simeq f(x) + \Delta x \, . \, f'(x)$, taking $x = .09$ and $\Delta x = -0.008$,

we get $f(0.09 - 0.008) = f(0.09) + (-0.008) f'(0.09)$

$$\Rightarrow \sqrt{0.082} = \sqrt{0.09} - 0.008 \, . \left(\frac{1}{2\sqrt{0.09}}\right) = 0.3 - \frac{0.008}{0.6}$$

$= 0.3 - 0.0133 = 0.2867$.

Example 8 Find the condition for the curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $xy = c^2$ to intersect orthogonally.

Solution Let the curves intersect at (x_1, y_1). Therefore,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$\Rightarrow$ slope of tangent at the point of intersection $(m_1) = \frac{b^2x_1}{a^2y_1}$

Again $xy = c^2 \Rightarrow x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x} \Rightarrow m_2=\frac{-y_1}{x_1}$.

For orthoganality, $m_1 \times m_2 = -1 \Rightarrow \frac{b^2}{a^2}=1$ or $a^2 - b^2 = 0$.

Example 9 Find all the points of local maxima and local minima of the function $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$.

Solution $f'(x) = -3x^3 - 24x^2 - 45x$

$= -3x(x^2 + 8x + 15) = -3x(x + 5)(x + 3)$

$f'(x) = 0 \Rightarrow x = -5,\ x = -3,\ x = 0$

$f''(x) = -9x^2 - 48x - 45$

$= -3(3x^2 + 16x + 15)$

$f''(0) = -45 < 0$. Therefore, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Therefore, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Therefore $x = -5$ is point of local maxima.

Example 10 Show that the local maximum value of $x+\frac{1}{x}$ is less than local minimum value.

Solution Let $y = x+\frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$,

$$\frac{dy}{dx} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

$$\frac{d^2y}{dx^2} = +\frac{2}{x^3}\text{, therefore } \frac{d^2y}{dx^2}(\text{at } x = 1) > 0 \text{ and } \frac{d^2y}{dx^2}(\text{at } x = -1) < 0.$$

Hence local maximum value of y is at $x = -1$ and the local maximum value = – 2.

Local minimum value of y is at $x = 1$ and local minimum value = 2.

Therefore, local maximum value (–2) is less than local minimum value 2.

Long Answer Type (L.A.)

Example 11 Water is dripping out at a steady rate of 1 cu cm/sec through a tiny hole at the vertex of the conical vessel, whose axis is vertical. When the slant height of water in the vessel is 4 cm, find the rate of decrease of slant height, where the vertical angle of the conical vessel is $\frac{\pi}{6}$.

Solution Given that $\frac{dv}{dt} = 1$ cm³/s, where v is the volume of water in the conical vessel.

From the Fig.6.2, $l = 4$cm, $h = l \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}l$ and $r = l \sin\frac{\pi}{6} = \frac{l}{2}$.

Therefore, $v = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\frac{l^2}{4}\frac{\sqrt{3}}{2}l = \frac{\sqrt{3}\pi}{24}l^3$.

$$\frac{dv}{dt} = \frac{\sqrt{3}\pi}{8} l^2 \frac{dl}{dt}$$

Therefore, $1 = \frac{\sqrt{3}\pi}{8} 16 . \frac{dl}{dt}$

$$\Rightarrow \qquad \frac{dl}{dt} = \frac{1}{2\sqrt{3}\pi} \text{ cm/s.}$$

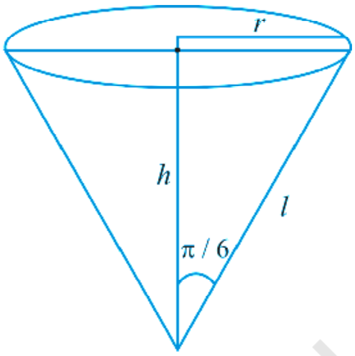

Fig. 6.2

Therefore, the rate of decrease of slant height = $\frac{1}{2\sqrt{3}\pi}$ cm/s.

Example 12 Find the equation of all the tangents to the curve $y = \cos (x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.

Solution Given that $y = \cos (x + y) \Rightarrow \frac{dy}{dx} = -\sin (x + y) \left[1 + \frac{dy}{dx}\right]$ ...(i)

or $$\frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)}$$

Since tangent is parallel to $x + 2y = 0$, therefore slope of tangent $= -\frac{1}{2}$

Therefore, $-\frac{\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2} \Rightarrow \sin (x + y) = 1$ .... (ii)

Since $\cos (x + y) = y$ and $\sin (x + y) = 1 \Rightarrow \cos^2 (x + y) + \sin^2 (x + y) = y^2 + 1$

$\Rightarrow \quad 1 = y^2 + 1$ or $y = 0$.

Therefore, $\cos x = 0$.

Therefore, $x = (2n + 1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2...$

Thus, $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, but $x = \frac{\pi}{2}, x = \frac{-3\pi}{2}$ satisfy equation (ii)

Hence, the points are $\left(\frac{\pi}{2}, 0\right), \left(\frac{-3\pi}{2}, 0\right)$.

Therefore, equation of tangent at $\left(\frac{\pi}{2}, 0\right)$ is $y = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ or $2x + 4y - \pi = 0$, and

equation of tangent at $\left(\frac{-3\pi}{2}, 0\right)$ is $y = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$ or $2x + 4y + 3\pi = 0$.

Example 13 Find the angle of intersection of the curves $y^2 = 4ax$ and $x^2 = 4by$.

Solution Given that $y^2 = 4ax$...(i) and $x^2 = 4by$... (ii). Solving (i) and (ii), we get

$$\left(\frac{x^2}{4b}\right)^2 = 4ax \Rightarrow x^4 = 64\, ab^2 x$$

or $\quad x(x^3 - 64\, ab^2) = 0 \Rightarrow x = 0,\ x = 4a^{\frac{1}{3}} b^{\frac{2}{3}}$

Therefore, the points of intersection are (0, 0) and $\left(4a^{\frac{1}{3}} b^{\frac{2}{3}}, 4a^{\frac{2}{3}} b^{\frac{1}{3}}\right)$.

Again, $y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$ and $x^2 = 4by \Rightarrow \frac{dy}{dx} = \frac{2x}{4b} = \frac{x}{2b}$

Therefore, at (0, 0) the tangent to the curve $y^2 = 4ax$ is parallel to y-axis and tangent to the curve $x^2 = 4by$ is parallel to x-axis.

$\Rightarrow$ Angle between curves $= \frac{\pi}{2}$

At $\left(4a^{\frac{1}{3}} b^{\frac{2}{3}}, 4a^{\frac{2}{3}} b^{\frac{1}{3}}\right)$, m_1 (slope of the tangent to the curve (i)) $= 2\left(\frac{a}{b}\right)^{\frac{1}{3}}$

$= \frac{2a}{4a^{\frac{2}{3}} b^{\frac{1}{3}}} = \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}$, m_2 (slope of the tangent to the curve (ii)) $= \frac{4a^{\frac{1}{3}} b^{\frac{2}{3}}}{2b} = 2\left(\frac{a}{b}\right)^{\frac{1}{3}}$

Therefore, $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\dfrac{2\left(\dfrac{a}{b}\right)^{\frac{1}{3}} - \dfrac{1}{2}\left(\dfrac{a}{b}\right)^{\frac{1}{3}}}{1 + 2\left(\dfrac{a}{b}\right)^{\frac{1}{3}} \dfrac{1}{2}\left(\dfrac{a}{b}\right)^{\frac{1}{3}}}\right| = \dfrac{3a^{\frac{1}{3}}.b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}$

Hence, $\theta = \tan^{-1}\left(\dfrac{3a^{\frac{1}{3}}.b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}\right)$

Example 14 Show that the equation of normal at any point on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ is $4 (y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta$.

Solution We have $x = 3\cos\theta - \cos^3\theta$

Therefore, $\dfrac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta = -3\sin\theta(1 - \cos^2\theta) = -3\sin^3\theta$.

$\dfrac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta\ (1 - \sin^2\theta) = 3\cos^3\theta$

$\dfrac{dy}{dx} = -\dfrac{\cos^3\theta}{\sin^3\theta}$. Therefore, slope of normal $= +\dfrac{\sin^3\theta}{\cos^3\theta}$

Hence the equation of normal is

$$y - (3\sin\theta - \sin^3\theta) = \frac{\sin^3\theta}{\cos^3\theta}[x - (3\cos\theta - \cos^3\theta)]$$

$\Rightarrow y\cos^3\theta - 3\sin\theta\cos^3\theta + \sin^3\theta\cos^3\theta = x\sin^3\theta - 3\sin^3\theta\cos\theta + \sin^3\theta\cos^3\theta$

$\Rightarrow y\cos^3\theta - x\sin^3\theta = 3\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

$$= \frac{3}{2}\sin 2\theta \cdot \cos 2\theta$$

$$= \frac{3}{4}\sin 4\theta$$

or $\quad 4\,(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta.$

Example 15 Find the maximum and minimum values of

$$f(x) = \sec x + \log \cos^2 x,\ 0 < x < 2\pi$$

Solution $f(x) = \sec x + 2\log \cos x$

Therefore, $f'(x) = \sec x \tan x - 2\tan x = \tan x\,(\sec x - 2)$

$$f'(x) = 0 \Rightarrow \tan x = 0 \text{ or } \sec x = 2 \text{ or } \cos x = \frac{1}{2}$$

Therefore, possible values of x are $x = 0$, or $x = \pi$ and

$$x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$

Again, $f''(x) = \sec^2 x\,(\sec x - 2) + \tan x\,(\sec x \tan x)$

$$= \sec^3 x + \sec x \tan^2 x - 2\sec^2 x$$

$$= \sec x\,(\sec^2 x + \tan^2 x - 2\sec x).$$ We note that

$f''(0) = 1\,(1 + 0 - 2) = -1 < 0$. Therefore, $x = 0$ is a point of maxima.

$f''(\pi) = -1\,(1 + 0 + 2) = -3 < 0$. Therefore, $x = \pi$ is a point of maxima.

$f''\left(\frac{\pi}{3}\right) = 2\,(4 + 3 - 4) = 6 > 0$. Therefore, $x = \frac{\pi}{3}$ is a point of minima.

$f''\left(\frac{5\pi}{3}\right) = 2\,(4 + 3 - 4) = 6 > 0$. Therefore, $x = \frac{5\pi}{3}$ is a point of minima.

Maximum Value of y at $x = 0$ is $\quad 1 + 0 = 1$

Maximum Value of y at $x = \pi$ is $\quad -1 + 0 = -1$

Minimum Value of y at $x = \frac{\pi}{3}$ is $\quad 2 + 2\log\frac{1}{2} = 2(1 - \log 2)$

Minimum Value of y at $x = \frac{5\pi}{3}$ is $\quad 2 + 2\log\frac{1}{2} = 2(1 - \log 2)$

Example 16 Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution Let ABCD be the rectangle of maximum area with sides AB = $2x$ and BC = $2y$, where C (x, y) is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in the Fig.6.3.

The area A of the rectangle is $4xy$ i.e. A = $4xy$ which gives $A^2 = 16x^2y^2 = s$ (say)

Therefore, $s = 16x^2\left(1-\frac{x^2}{a^2}\right).b^2 = \frac{16b^2}{a^2}(a^2x^2 - x^4)$

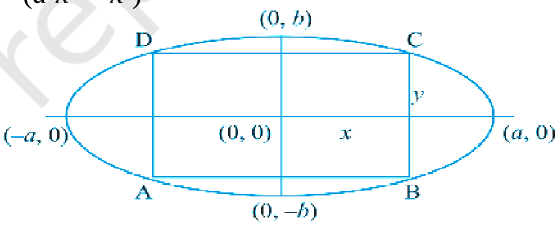

Fig. 6.3

$\Rightarrow \quad \frac{ds}{dx} = \frac{16b^2}{a^2}.[2a^2x - 4x^3].$

Again, $\frac{ds}{dx} = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$ and $y = \frac{b}{\sqrt{2}}$

Now, $\frac{d^2s}{dx^2} = \frac{16b^2}{a^2}[2a^2 - 12x^2]$

At $x = \frac{a}{\sqrt{2}}, \frac{d^2s}{dx^2} = \frac{16b^2}{a^2}[2a^2 - 6a^2] = \frac{16b^2}{a^2}(-4a^2) < 0$

Thus at $x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}}, s$ is maximum and hence the area A is maximum.

Maximum area $= 4.x.y = 4 . \frac{a}{\sqrt{2}} . \frac{b}{\sqrt{2}} = 2ab$ sq units.

Example 17 Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$, on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution $f(x) = \sin 2x - x$

$\Rightarrow \quad f'(x) = 2 \cos 2x - 1$

Therefore, $f'(x) = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x$ is $\frac{-\pi}{3}$ or $\frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6}$ or $\frac{\pi}{6}$

$$f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = \frac{\pi}{2}$$

$$f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{2\pi}{6}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$f\left(\frac{\pi}{2}\right) = \sin(\pi) - \frac{\pi}{2} = -\frac{\pi}{2}$$

Clearly, $\frac{\pi}{2}$ is the greatest value and $-\frac{\pi}{2}$ is the least.

Therefore, difference $= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Example 18 An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Solution Let ABC be an isosceles triangle inscribed in the circle with radius a such that AB = AC.

AD = AO + OD = $a + a\cos 2\theta$ and BC = 2BD = $2a\sin 2\theta$ (see fig. 16.4)

Therefore, area of the triangle ABC i.e. $\Delta = \frac{1}{2}$BC . AD

$$= \frac{1}{2} 2a\sin 2\theta \,.\, (a + a\cos 2\theta)$$

$$= a^2 \sin 2\theta\,(1 + \cos 2\theta)$$

$$\Rightarrow \quad \Delta = a^2\sin 2\theta + \frac{1}{2}a^2\sin 4\theta$$

Therefore, $\frac{d\Delta}{d\theta} = 2a^2\cos 2\theta + 2a^2\cos 4\theta$

$$= 2a^2(\cos 2\theta + \cos 4\theta)$$

$$\frac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta = -\cos 4\theta = \cos(\pi - 4\theta)$$

Therefore, $2\theta = \pi - 4\theta \Rightarrow \theta = \frac{\pi}{6}$

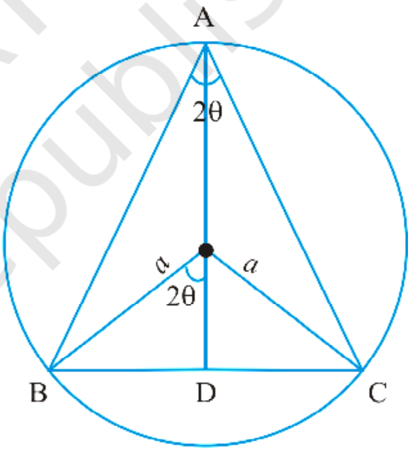

Fig. 6.4

$$\frac{d^2\Delta}{d\theta^2} = 2a^2(-2\sin 2\theta - 4\sin 4\theta) < 0 \text{ (at } \theta = \frac{\pi}{6}\text{)}.$$

Therefore, Area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the following Examples 19 to 23.

Example 19 The abscissa of the point on the curve $3y = 6x - 5x^3$, the normal at which passes through origin is:

(A) 1 (B) $\frac{1}{3}$ (C) 2 (D) $\frac{1}{2}$

Solution Let (x_1, y_1) be the point on the given curve $3y = 6x - 5x^3$ at which the normal passes through the origin. Then we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 - 5x_1^2$. Again the equation of the normal at (x_1, y_1) passing through the origin gives $2 - 5x_1^2 = \frac{-x_1}{y_1} = \frac{-3}{6 - 5x_1^2}$.

Since $x_1 = 1$ satisfies the equation, therefore, Correct answer is (A).

Example 20 The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$

(A) touch each other (B) cut at right angle

(C) cut at an angle $\frac{\pi}{3}$ (D) cut at an angle $\frac{\pi}{4}$

Solution From first equation of the curve, we have $3x^2 - 3y^2 - 6xy \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = (m_1)$$ say and second equation of the curve gives

$$6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = (m_2) \text{ say}$$

Since $m_1 \cdot m_2 = -1$. Therefore, correct answer is (B).

Example 21 The tangent to the curve given by $x = e^t \cdot \cos t$, $y = e^t \cdot \sin t$ at $t = \frac{\pi}{4}$ makes with x-axis an angle:

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution $\frac{dx}{dt} = -e^t \cdot \sin t + e^t \cos t$, $\frac{dy}{dt} = e^t \cos t + e^t \sin t$

Therefore, $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{\cos t + \sin t}{\cos t - \sin t} = \frac{\sqrt{2}}{0}$ and hence the correct answer is (D).

Example 22 The equation of the normal to the curve $y = \sin x$ at (0, 0) is:

(A) $x = 0$ (B) $y = 0$ (C) $x + y = 0$ (D) $x - y = 0$

Solution $\frac{dy}{dx} = \cos x$. Therefore, slope of normal $= \left(\frac{-1}{\cos x}\right)_{x=0} = -1$. Hence the equation of normal is $y - 0 = -1(x - 0)$ or $x + y = 0$

Therefore, correct answer is (C).

Example 23 The point on the curve $y^2 = x$, where the tangent makes an angle of $\frac{\pi}{4}$ with x-axis is

(A) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) (4, 2) (D) (1, 1)

Solution $\frac{dy}{dx} = \frac{1}{2y} = \tan\frac{\pi}{4} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

Therefore, correct answer is B.

Fill in the blanks in each of the following Examples 24 to 29.

Example 24 The values of a for which $y = x^2 + ax + 25$ touches the axis of x are______.

Solution $\frac{dy}{dx} = 0 \Rightarrow 2x + a = 0$ i.e. $x = -\frac{a}{2}$,

Therefore, $\frac{a^2}{4} + a\left(-\frac{a}{2}\right) + 25 = 0 \quad \Rightarrow \quad a = \pm 10$

Hence, the values of a are $\pm$ 10.

Example 25 If $f(x) = \frac{1}{4x^2 + 2x + 1}$, then its maximum value is _______.

Solution For f to be maximum, $4x^2 + 2x + 1$ should be minimum i.e.

$4x^2 + 2x + 1 = 4(x + \frac{1}{4})^2 + \left(1 - \frac{1}{4}\right)$ giving the minimum value of $4x^2 + 2x + 1 = \frac{3}{4}$.

Hence maximum value of $f = \frac{4}{3}$.

Example 26 Let f have second deriative at c such that $f'(c) = 0$ and $f''(c) > 0$, then c is a point of ______.

Solution Local minima.

Example 27 Minimum value of f if $f(x) = \sin x$ in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is _____.

Solution –1

Example 28 The maximum value of $\sin x + \cos x$ is _____.

Solution $\sqrt{2}$.

Example 29 The rate of change of volume of a sphere with respect to its surface area, when the radius is 2 cm, is______.

Solution 1 cm^3/cm^2

$$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2, \; s = 4\pi r^2 \Rightarrow \frac{ds}{dr} = 8\pi r \Rightarrow \frac{dv}{ds} = \frac{r}{2} = 1 \text{ at } r = 2.$$

6.3 EXERCISE

Short Answer (S.A.)

1. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is propotional to the surface. Prove that the radius is decreasing at a constant rate.

2. If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

3. A kite is moving horizontally at a height of 151.5 meters. If the speed of kite is 10 m/s, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of boy is 1.5 m.

4. Two men A and B start with velocities *v* at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being seperated..

5. Find an angle θ, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

6. Find the approximate value of $(1.999)^5$.

7. Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

8. A man, 2m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?

9. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and L = 200 $(10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

10. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

11. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

12. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

13. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

14. Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

15. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

16. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

17. Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

18. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

19. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$, touches the curve $y = b \cdot e^{\frac{-x}{a}}$ at the point where the curve intersects the axis of y.

20. Show that $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing in **R**.

21. Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in **R**.

22. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

23. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also find the maximum slope.

24. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Long Answer (L.A.)

25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

26. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

27. A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs 300/- per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Re 1/- one subscriber will discontinue the service. Find what increase will bring maximum profit?

28. If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

29. An open box with square base is to be made of a given quantity of card board of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

30. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

32. AB is a diameter of a circle and C is any point on the circle. Show that the area of Δ ABC is maximum, when it is isosceles.

33. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and bottom costs Rs 5/cm^2 and the material for the sides costs Rs 2.50/cm^2 . Find the least cost of the box.

34. The sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

Objective Type Questions

Choose the correct answer from the given four options in each of the following questions 35 to 39:

35. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:

(A) 10 cm^2/s (B) $\sqrt{3}$ cm^2/s (C) $10\sqrt{3}$ cm^2/s (D) $\frac{10}{3}$ cm^2/s

36. A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:

(A) $\frac{1}{10}$ radian/sec (B) $\frac{1}{20}$ radian/sec (C) 20 radian/sec

(D) 10 radian/sec

37. The curve $y = x^{\frac{1}{5}}$ has at (0, 0)

(A) a vertical tangent (parallel to y-axis)

(B) a horizontal tangent (parallel to x-axis)

(C) an oblique tangent

(D) no tangent

38. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(A) $3x - y = 8$ (B) $3x + y + 8 = 0$

(C) $x + 3y \pm 8 = 0$ (D) $x + 3y = 0$

39. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is:

(A) 1 (B) 0 (C) – 6 (D) .6

40. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y

(A) .32 (B) .032 (C) 5.68 (D) 5.968

41. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is:

(A) $x + 5y = 2$ (B) $x - 5y = 2$

(C) $5x - y = 2$ (D) $5x + y = 2$

42. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are:

(A) (2, –2), (–2, –34) (B) (2, 34), (–2, 0)

(C) (0, 34), (–2, 0) (D) (2, 2), (–2, 34)

43. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at:

(A) (0, 1) (B) $\left(-\frac{1}{2}, 0\right)$ (C) (2, 0) (D) (0, 2)

44. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is:

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{-6}{7}$ (D) -6

45. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

46. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is:

(A) $[-1, \infty)$ (B) $[-2, -1]$ (C) $(-\infty, -2]$ (D) $[-1, 1]$

47. Let the $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + \cos x$, then f :

(A) has a minimum at $x = \pi$ (B) has a maximum, at $x = 0$

(C) is a decreasing function (D) is an increasing function

48. $y = x(x - 3)^2$ decreases for the values of x given by :

(A) $1 < x < 3$ (B) $x < 0$ (C) $x > 0$ (D) $0 < x < \frac{3}{2}$

49. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(A) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (B) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(C) decreasing in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (D) decreasing in $\left[0, \frac{\pi}{2}\right]$

50. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$

(A) $\sin 2x$ (B) $\tan x$ (C) $\cos x$ (D) $\cos 3x$

51. The function $f(x) = \tan x - x$

(A) always increases (B) always decreases

(C) never increases (D) sometimes increases and sometimes decreases.

52. If x is real, the minimum value of $x^2 - 8x + 17$ is

(A) –1 (B) 0 (C) 1 (D) 2

53. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is

(A) 126 (B) 0 (C) 135 (D) 160

54. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(A) two points of local maximum (B) two points of local minimum

(C) one maxima and one minima (D) no maxima or minima

55. The maximum value of $\sin x \,.\, \cos x$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) $2\sqrt{2}$

56. At $x = \frac{5\pi}{6}$, $f(x) = 2 \sin 3x + 3 \cos 3x$ is:

(A) maximum (B) minimum

(C) zero (D) neither maximum nor minimum.

57. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(A) 0 (B) 12 (C) 16 (D) 32

58. $f(x) = x^x$ has a stationary point at

(A) $x = e$ (B) $x = \frac{1}{e}$ (C) $x = 1$ (D) $x = \sqrt{e}$

59. The maximum value of $\left(\frac{1}{x}\right)^x$ is:

(A) e (B) e^e (C) $e^{\frac{1}{e}}$ (D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$

Fill in the blanks in each of the following Exercises 60 to 64:

60. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point_____.

61. The equation of normal to the curve $y = \tan x$ at (0, 0) is ________.

62. The values of a for which the function $f(x) = \sin x - ax + b$ increases on **R** are ______.

63. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, $x > 0$, decreases in the interval _______.

64. The least value of the function $f(x) = ax + \dfrac{b}{x}$ $(a > 0, b > 0, x > 0)$ is ______.